“α-Degree Closures for Graphs”

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\textbf{\textit{\Large$\alpha$—Degree Closures for Graphs}}

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\textbf{Abstract}

Bondy and Chvátal [7] introduced a general and unified approach to a variety of graph-theoretic problems. They defined the \textit{k-closure} $C_k(G)$, where \textit{k} is a positive integer, of a graph $G$ of order \textit{n} as the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices \textit{a, b} satisfying the condition $C(a, b) : d(a) + d(b) \geq k$. For many properties \textit{P}, they found a suitable \textit{k} (depending on \textit{P} and \textit{n}) such that $C_k(G)$ has property \textit{P} if and only if $G$ does. For instance, if \textit{P} is the hamiltonian property, then \textit{k} = \textit{n}.

In [3], we proved that $C(a, b)$ can be replaced by $d(a) + d(b) + |Q(G)| \geq \textit{k}$, where $Q(G)$ is a well-defined subset of vertices nonadjacent to \textit{a, b}.

In [4], we proved that, for a $(2 + \textit{k} - \textit{n})$-connected graph, $C(a, b)$ can be replaced by $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq \textit{k}$, where $\varepsilon_{ab}$ is a well defined binary variable and $\delta_{ab}$ is the minimum degree over all vertices distinct from \textit{a, b} and non adjacent to them. The condition on connectivity is a necessary one.

In this paper we show that $C(a, b)$ can be replaced by the condition $d(a) + d(b) + (\overline{\alpha_{ab}} - \alpha_{ab}) \geq \textit{k}$, where $\overline{\alpha_{ab}}$ and $\alpha_{ab}$ are respectively the order and the independence number of the subgraph $G - N(a) \cup N(b)$.

All these three last conditions are uncomparable, unique and well defined. Moreover any hamiltonian cycle in $C_n(G)$ can be transformed into a hamiltonian cycle in the original graph within a polynomial time. However, unlike the conditions given in [3] and [4], the condition $(\overline{\alpha_{ab}} - \alpha_{ab})$ cannot be computed in polynomial time. By giving suitable upper bounds of $\alpha_{ab}$ (or lower bounds of $(\overline{\alpha_{ab}} - \alpha_{ab})$) we satisfy this last nice property. In doing so, we surprisingly obtain a result of [8] as an easy Corollary.

\textbf{Key words:} Closure, Degree Closure, Neighborhood Closure, Dual Closure, Stability, Hamiltonicity, Cyclability, Degree Sequence, Matching Number, $k$-Leaf-Connected.
1 Introduction

Let $G = (V, E)$ be a finite simple graph of order $n$, connectivity $\kappa(G)$. Ore [7] proved that $G$ is hamiltonian if the condition $d(a) + d(b) \geq n$ is satisfied by any pair $(a, b)$ of nonadjacent vertices. Later, Bondy and Chvátal [7] observed that $G$ is in fact hamiltonian if and only if $G + ab$ is hamiltonian. This observation motivated the introduction of the concept of the $k$–closure $C_k(G)$ of $G$, for a given positive integer $k$. The graph $C_k(G)$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $k$. This graph is unique and polynomially obtained from $G$. For a number of various properties of a graph $G$ on $n$ vertices, they showed that it is possible to find a suitable integer $k$, such that if $G$ has property $P(k)$, so does $C_k(G)$. For instance, if $P$ is the hamiltonian property, then $k = n$.

Starting from the main result obtained in [2] we improved the condition $P(k) : d(a) + d(b) \geq k$ in two directions:

In [3], $P(k)$ becomes: $d(a) + d(b) + |Q(G)| \geq k$, where $Q(G)$ is a well-defined subset of vertices nonadjacent to $a$, $b$. The corresponding condition is named “$\beta$–dec” for $\beta$–degree closure condition.

In [4], for a $(2 + k - n)$-connected graph, $P(k)$ becomes: $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq k$, where $\varepsilon_{ab}$ is a well defined binary variable and $\delta_{ab}$ is the minimum degree over all vertices distinct from $a, b$ and non adjacent to them. The corresponding condition is named “$\beta$–nccl” for $\beta$–neighborhood closure condition. The condition on connectivity is not a real constraint since it is a necessary condition.

In this paper, we use a relaxation of the main result given in [1] to obtain another improvement of $P(k)$. The new condition is: $d(a) + d(b) + |\sigma_{ab} - \alpha_{ab}| \geq k$, where $\sigma_{ab}$ and $\alpha_{ab}$ are respectively the order and the independence number of the subgraph $G - N(a) \cup N(b)$. We shall refer to it as “$\alpha$–dec” for $\alpha$–degree closure condition.

To state the new results and to relate them to existing ones, we need some preliminary definitions and notations.

2 Definitions and notations

We use Bondy and Murty [9] for terminology and notation not defined here and consider simple graphs only. Let $G = (V, E)$ be a graph of order $n \geq 3$. The set of neighbors of a vertex $v \in V$ is denoted $N_G(v)$ and $d_G(v) = |N_G(v)|$ is the degree of $v$. If $A$ is a subset of $V$, $G[A]$ will denote the subgraph induced by $A$.

Let $C$ be a cycle in $G$, in which a direction of traversing it is given. For $u \in V(C)$, $u^+$ (resp. $u^-$) denotes its successor (resp. predecessor) on $C$. More generally, if $A \subseteq V$ then $A^+ := \{u \in C \mid u^- \in A\}$ and $A^- := \{u \in C \mid u^+ \in A\}$. Given vertices $a, b$ of $C$ let $C[a, b]$ denote the subgraph of $C$ from $a$ to $b$ in the chosen direction. We shall write $C(a, b)$ or $C(a, b)$ or $C(a, b)$ if $a, b$ or $(a$ and $b)$ are respectively excluded. The same notation will be adopted if we consider a path $P$ (where the direction of traversing it is assumed) instead of a cycle $C$. 

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Paths and cycles in $G = (V, E)$ are considered as subgraphs and for simplicity we use the same notation to mean a subgraph, its vertex set or its edge set.

The concept of vine [6] plays a central role for our proofs. A vine on a path $\pi := \pi[a, b]$ is a set $\mathcal{P} := \{\pi_i[x_i, y_i] \mid 1 \leq i \leq m\}$ of internally disjoint paths such that
\begin{itemize}
  \item[(a)] $\pi_i \cap \pi = \{x_i, y_i\}$
  \item[(b)] $a = x_1 \prec x_2 \prec y_1 \preceq x_3 \preceq y_2 \preceq x_4 \prec \ldots \preceq x_m \prec y_m = b$ on $\pi$
\end{itemize}
(here $u \prec v$ (resp. $u \preceq v$) means that $u$ precedes $v$ on $P$ (resp. possibly $u = v$) where $\pi$ is oriented from $a$ to $b$).

With each vine $\mathcal{P}$ on a path $\pi[a, b]$ is associated a constrained cycle $C_{ab} := \sum_{i=1}^{m} C_i$, where $C_i := \pi[x_i, y_i]$, $1 \leq i \leq m$ and the addition of edges in $\sum_{i=1}^{m} C_i$ is taken modulo $2$. In this paper the paths $\pi_i[x_i, y_i]$ are in fact edges because we shall only focusing on a Hamiltonian path $\pi$ instead of any path.

Let $(a, b)$ be a pair of nonadjacent vertices, $x$ be any vertex not adjacent to $a$ and $b$ and $k$ be a positive integer. Then we associate

\[
\begin{array}{ll}
  (a) & G_{ab} := G - N_G(a) \cup N_G(b), \ T_{ab}(G) := V(G_{ab}) \setminus \{a, b\} \\
  (b) & \bar{G}_{ab}(G) := |G_{ab}| = 2 + |T_{ab}(G)|, \ \alpha_{ab} := \alpha(G_{ab}), \ \nu_{ab} := \nu(G_{ab}) \\
  (c) & \Delta_{ab}(G) := \max \{d_G(x) \mid x \in T_{ab}(G)\}, \ \delta_{ab}(G) := \min \{d_G(x) \mid x \in T_{ab}(G)\} \\
  (d) & \sigma_{ab}(G) := d_G(a) + d_G(b), \ \gamma_{ab}(G) := |N_G(a) \cup N_G(b)| \\
  (e) & \lambda_{ab}(G) := |N_G(a) \cap N_G(b)|.
\end{array}
\]

Note that $G_{ab}$ is disconnected since $a, b$ are isolated vertices and $\nu(G_{ab})$ is the matching number of $G_{ab}$. For a given Hamiltonian path $\mu$, let the vertices be ordered so that $i < j$ implies that $i$ appears before $j$ on the path $\mu$. Traversed from $a$ to $b$. Let a directed graph $\bar{G}$ be produced from $G$ by designating a direction to arc $ij$ of $G$ from $i$ to $j$ whenever $i < j$. The $a$ to $b$ vertex connectivity of $\bar{G}$ is denoted $h_{ab}^G$. Dirac [10] proved that a vine with two paths exists on any path in a two-connected graph. In that case, these two paths satisfy the constraint on $h_{ab}^G$. In other words, $h_{ab}^G \geq 2$ holds for any 2-connected graph.$\bar{G}$.

In [1], we proved:

**Theorem 1** Let $G$ be a graph of order $n \geq 3$. If $\alpha_{ab} \leq h_{ab}^G$ then $G$ is Hamiltonian if and only if $G + ab$ is Hamiltonian.

In [2] we conjectured the following.

**Conjecture 1** Let $G$ be a $\kappa$-connected graph of order $n \geq 3$. If $\alpha_{ab} \leq \max(\kappa, \lambda_{ab})$ then $G$ is Hamiltonian if and only if $G + ab$ is Hamiltonian.

The condition $\alpha_{ab} \leq \max(\kappa, \lambda_{ab})$ will be referred to as the "$\alpha - cc$" for $\alpha$-closure condition. This new condition admits two incomparable relaxations.

- Going beyond our result in [1], we treat the case $\max(\kappa, \lambda_{ab}) = \lambda_{ab}$ in this paper. In particular we get the main result of [8] as an easy corollary. The
corresponding condition of this case, that is, \( \alpha_{ab} \leq \lambda_{ab} \), will be referred as the \( \alpha \)-degree closure condition (\( \alpha - \text{dcc} \)). This condition involves the degree sum of \((a, b)\) since \( \alpha_{ab} \leq \lambda_{ab} \Leftrightarrow \sigma_{ab} + (\bar{\sigma}_{ab} - \alpha_{ab}) \geq n \).

- In another paper in preparation [5] we consider the conjectured part of the condition, that is \( \max(\kappa, \lambda_{ab}) = \kappa \). This will be referred as the alpha-neighborhood closure condition (\( \alpha - \text{ncc} \)). Particular cases \( \kappa = 2, 3 \) will be proved and a particular condition treated in [8] will be improved.

Following Bondy and Chvátal ([7]), we define:

**Definition 1** Let \( P \) be a property defined for all graphs \( G \) of order \( n \) and let \( k \) be an integer. Let \( a, b \) be two nonadjacent vertices satisfying the condition

\[
P(k) : \alpha_{ab}(G) \leq \lambda_{ab} + n - k \Leftrightarrow \sigma_{ab}(G) + (\bar{\sigma}_{ab} - \alpha_{ab}) \geq k.
\]


Then \( P \) is \( k \)-alpha degree stable if whenever \( G + ab \) has property \( P \) and \( P(k) \) holds then \( G \) itself has property \( P \). We simply denote by \( dC_k(G) \) the associated (\( \alpha \)-degree closure).

The graph \( dC_k(G) \) is then obtained from \( G \) by recursively joining pairs of nonadjacent vertices \( a, b \) for which (*) holds until no such pair remains. The equivalence in (*) comes from the equalities \( \sigma_{ab} = d(a) + d(b) = \gamma_{ab} + \lambda_{ab} \) and \( \bar{\sigma}_{ab} + \gamma_{ab} = n \). For the very particular case where \( T_{ab} \) is an independent set, (*) reduces to Bondy-Chvátal’s known closure condition. The statement below is an easy adaptation of Proposition 2.1 in [7].

**Proposition 1** If \( P \) is \( k \)-alpha degree stable and \( dC_k(G) \) has property \( P \) then \( G \) itself has property \( P \).

In this paper, we investigate the stability of a number of properties of graphs which remain in any super-graph of \( G \) (a graph obtained from \( G \) by addition of edges). Most of these properties are studied in [7]. We also provide new properties. Throughout let \((a, b)\) be a pair of nonadjacent vertices of a graph \( G \) satisfying the condition (*) for a given positive integer \( k \). For each one of the considered properties \( P \) we fix \( k \) so that \( G \) has properties \( P \) whenever \( G + ab \) does. Below is a key-lemma for the remaining of the paper.

**Lemma 1** Let \( \pi[a, b] \) be a hamiltonian \( a - b \) path. If \( \alpha_{ab} \leq \lambda_{ab} \) then \( G \) is hamiltonian.

**Proof.** A proof by induction is already given in [1]. Here we provide an alternative constructive one which has its own interest. By contradiction we assume \( G \) nonhamiltonian.

Set \( W := \{w_i \in N(a) \cap N(b) | i = 1, \ldots, \lambda_{ab}\} \) and \( W_j := \pi(w_{j}, w_{j+1}) \) for \( j = 1, \ldots, \lambda_{ab} - 1 \). It is clear that the vertices of \( W \) cannot be consecutive on \( \pi \) and
$W_i \subseteq T$ holds for all $i$ since otherwise we have an obvious Hamiltonian cycle. This implies that $T \neq \emptyset$ and $\alpha_{ab} \geq 3$. This in turn implies $\lambda_{ab} \geq 3$. Therefore we have $w_1 \prec w_2 \prec \ldots \prec w_{\lambda_{ab}}$. Within each $W_i$, choose a subinterval $D_i := \pi(b_i, a_i)$ such that $b_i \in N(b)$, $a_i \in N(a)$ and $D_i \subseteq T$. Such a subpath exists as we always may choose $b_i = w_i$, $a_i = w_{i+1}$. Otherwise we have $w_i \prec b_i \prec a_i \prec w_{i+1}$ for $i = 1, \ldots, \lambda_{ab}$.

We now define a set $R := \{r_1, \ldots, r_{\lambda_{ab} - 1}\}$ as follows:

(i) $r_1 := a_1^1$. Clearly $r_1 \in W_1 \neq \emptyset$ exists.

(ii) $r_2 \in W_2 \setminus [N(r_1)]$ is chosen so that $r_2^+_1 \in N(a) \cup N(r_1)$. If $W_2 \cap N(r_1) = \emptyset$ we set $r_2 := a_2$. Note that $r_1 b_2^+ \notin E$ since otherwise the constrained cycle of the vine $P := \{aa_1, r_1 b_2^+, b_2 b\}$ is Hamiltonian. Thus $r_2 \neq b_2$ exists and $\{r_1, r_2\}$ is an independent set.

(iii) $r_3 \in W_3 \setminus [N(r_1) \cup N(r_2)]$ is chosen so that $r_3^+_2 \in N(a) \cup N(r_1) \cup N(r_2)$. If $W_3 \cap [N(r_1) \cup N(r_2)] = \emptyset$ we set $r_3 = a_3$. Note that $r_1 b_3^+ \notin E$ since otherwise the constrained cycle of either the vine $P := \{aa_2, r_2 b_3^+, b_3 b\}$ is Hamiltonian if $r_2 = a_2^+$ or the vine $P := \{aa_1, r_1 r_2^+, r_2 b_3^+, b_3 b\}$ is Hamiltonian if $r_2 \prec a_2^+$ and $r_1 = a_1^+$. We observe that $r_3$ exists and $\{r_1, r_2, r_3\}$ is an independent set.

(iv) We continue this way and for $3 < i \leq \lambda_{ab} - 1$ we choose $r_i \in W_i \setminus [r_{i-1}^{-1} N(r_{i-1})]$ such that $r_i^+_1 \in N(a) \cup [r_{i-1}^{-1} N(r_{i-1})]$. If $W_i \cap [r_{i-1}^{-1} N(r_{i-1})] = \emptyset$ we set $r_i = a_i^+$. Following the above method we reach the conclusion that $R := \{r_1, \ldots, r_{\lambda_{ab}}\}$ is an independent set. This is a contradiction to the hypothesis since then $\alpha_{ab} \geq \{|a, b| \cup R| > \lambda_{ab}$.]

We would like to point out that the proof of the above Lemma shows that one can find a Hamiltonian cycle in G in polynomial time if we know one in $dC_k(G)$. However the construction of the closure itself cannot be found in polynomial time as it is well known to be a hard problem to compute the independence number $\alpha_{ab}$. This is why we provide in section 4 an alternative closure condition which is a relaxation of $P(k)$.

Throughout, $S \subseteq V$ denotes a subset with $s$ vertices.

3 Main results

Theorem 2 The property of being Hamiltonian is $n$-degree stable.

Proof. Consequence of Lemma 1. ■

The graph $G$ is $S$-Hamiltonian, $s \leq n - 3$, if it remains Hamiltonian whenever some or all vertices of $S$ are removed. We simply say that it is $s$-Hamiltonian if we are only interested by the number $s$ instead of the set $S$ of vertices.

Theorem 3 The property of being $S$-Hamiltonian is $(n + s)$-degree stable.
Proof. For some $W \subseteq S$, set $H := G - W$. By the hypothesis $\alpha_{ab}(G) \leq \lambda_{ab}(G) - s$. Clearly $\alpha_{ab}(H) \leq \alpha_{ab}(G)$ and $\lambda_{ab}(G) \leq \lambda_{ab}(H) + |W|$. Thus $\alpha_{ab}(H) \leq \alpha_{ab}(G) \leq \lambda_{ab}(G) - s \leq \lambda_{ab}(H) + |W| - s$. It follows that $\alpha_{ab}(H) \leq \lambda_{ab}(H)$ since $|W| - s \leq 0$. Therefore $H$ is hamiltonian by Theorem 2. Note that this property is $(n + s - 1)$ - degree stable if $S$ is not an independent set, in which case $\pi_{ab} - \alpha_{ab} \geq 1$ The proof is now complete. \[\square\]

The subgraph $G[S]$ is hamiltonian if $G$ is $V \setminus S$-hamiltonian. Applying Theorem 3 we obtain:

Theorem 4 The property "$G[S]$ is hamiltonian" is $(2n - s)$-degree stable.

We say that $G$ is $S$-cyclic (S- traceable resp.) if it contains a cycle $C$ (a path resp.) with all vertices of $S$.

Theorem 5 The property "$G$ is $S$-cyclic" is $n$-degree stable.

Proof. Suppose that $(G + ab)$ contains a cycle $C$ such that $S \subseteq V(C)$ but $G$ does not. Then $a, b$ are connected by a path $\pi := a_1 \ldots a_p$ with $a = a_1, b = a_p$, $n \geq p \geq s$. Set $H := G[V(\pi)]$ and $W := V \setminus V(H)$. Clearly $N(a) \cap N(b) \subseteq V(\pi)$ since otherwise $H$ is hamiltonian. Obviously $\alpha_{ab}(H) \leq \alpha_{ab}(G)$. By the hypothesis $\alpha_{ab}(G) \leq \lambda_{ab}(G)$. Thus $\alpha_{ab}(H) \leq \alpha_{ab}(G) \leq \lambda_{ab}(G) = \lambda_{ab}(H)$. Therefore $H$ is hamiltonian by Theorem 2. \[\square\]

A caterpillar is a particular tree which results in a path when its leaves are removed. The spine of the caterpillar is the longest path of it. The graph $G$ is called $S$-caterpillar spannable if it has a spanning tree that is a caterpillar, whose leaves are the vertices of $S := \{x_1, \ldots, x_s\}$. Suppose that the spine is an $[x_1, x_2]$-path. Let $G'$ be a graph obtained from $G$ by adding a new vertex, $v$ say, that is joined to $x_1$ and $x_2$. Then $G$ is $S$-caterpillar spannable if $G'$ is $(S - \{x_1, x_2\})$-hamiltonian. Applying Theorem 3 to the graph $G'$ we obtain

Theorem 6 Let $S \subseteq V(G)$ with $s$ vertices, $2 \leq s < n$. Then the property "$G$ is $S$-caterpillar spannable" is $(n + s - 1)$-degree stable.

A set $F \subseteq E$ of edges such that the components of the graph $(V, F)$ are vertex disjoint paths is called $F$-cyclic (or $|F|$-edge-hamilton) if there exists a cycle that contains $F$. It is $F$-traceable if there exists a path that contains $F$. Applying Theorem 2 to the graph obtained from $G$ by subdividing each edge in $F$ into two, we obtain

Theorem 7 The property "$G$ is $F$-cyclic with $|F| \leq n - 3$", is $(n + |F|)$-degree stable.

A graph $G$ is defined to be $|F|$-Hamilton-connected if for each pair $(x, y)$ of vertices there is a hamiltonian path with endpoints $x, y$ that contains $F$. Now $G$ must be $(F \cup xy)$-cyclic and using Theorem 7 we obtain

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Theorem 8 The property "G is F-Hamilton-connected with |F| ≤ n − 4", is $(n + |F| + 1)$-degree stable.

Let $sK_2$ be an $s$-matching, that is, a subgraph with $s$ independent edges.

Theorem 9 Let $n, s$ be positive integers with $s ≤ \frac{n}{2}$. Then the property of containing $sK_2$ is $(2s - 1)$-degree stable.

Proof. If $G + ab$ contains an $sK_2$ but $G$ does not, then there exists an $(s-1)$-matching $\{a_1b_1, ..., a_{s-1}b_{s-1}\}$ in $G$ and an $s$-matching in $G + ab$. For $i \in [1, s - 1]$ we set

$$A := \{a_i\}, B := \{b_i\}, D := V \setminus (A \cup B \cup \{a, b\}),$$

$$M := \{a_ib_i| i \in [1, s - 1]\}, M_i = \{a_i, b_i\} and m_i := |N_{M_i}(a) \cup N_{M_i}(b)|.$$

We label the vertices of $A, B$ so that $a_i \in N(a) \cup N(b)$ whenever $m_i ≥ 1$. An $M$-augmenting path is a path with an even number of vertices, unsaturated endpoints in $D \cup \{a, b\}$ and whose edges are alternatively in $E - M$ and $M$. To avoid contradiction, we obviously assume that $G$ contains no $M$-augmenting path. Moreover $D \cup \{a, b\}$ must be an independent set since otherwise there is an $s$-matching in $G$. We shall assume $\alpha_{ab} ≠ \overline{\alpha}_{ab}$ (that is $T_{ab}$ is not an independent set), by Bondy and Chvátal’s result [7].

To distinguish all possible configurations we define the following independent sets: $J_0 := \{i|m_i = 0\}, J_{11} := \{i|m_i = 1 and |N(a_i) \cap \{a, b\}| = 1\}, J_{12} := \{i|m_i = 1 and |N(a_i) \cap \{a, b\}| = 2\}$ and $J_2 := \{i|m_i = 2\}$. If $j \in J_2$ then $d_{M_j}(a) + d_{M_j}(b) = 2$ and either $d_{M_j}(a) = 2$ or $d_{M_j}(b) = 2$ for if $aa_j, bb_j \in E$ then $aa_jb_j$ is an $M$-augmenting path. These sets form a partition of $J := J_0 \cup J_{11} \cup J_{12} \cup J_2$. We note that

$$\sigma_{ab} = |J_{11}| + 2(|J_{12}| + |J_2|), \quad s = |J| + 1 \quad (1)$$

and $$\overline{\sigma}_{ab} = 2 + |J_{11}| + |J_{12}| + 2|J_0| + |D|. \quad (2)$$

By the hypothesis $\sigma_{ab} + \overline{\sigma}_{ab} = 2 + 2|J| + |J_{12}| + |D| ≥ 2s - 1 + \alpha_{ab}$. On the other hand, we prove that $\alpha_{ab} ≥ 2 + |J_{12}| + |D|$. It suffices to prove that $\{a, b\} \cup \{b_i| i \in J_{12}\} \cup D$ is an independent set. We already know that $\{a, b\} \cup D$ is independent. If $D \neq \emptyset$, choose $x \in D$ and suppose $xb_j \in E$ with $1 \in J_{12}$. Then $a_j \in N(a) \cap N(b)$ and $aa_jb_jx$ is an $M$-augmenting path. It remains to prove that $bh_{12} \notin E$ if $1, 2 \in J_{12}$. Otherwise $aa_1b_1b_2a_2b$ is an $M$-augmenting path. Finally we have $\sigma_{ab} + \overline{\sigma}_{ab} ≥ 2s - 1 + \alpha_{ab} ≥ 2s - 1 + 2 + |J_{12}| + |D|$, that is $2|J| ≥ 2s - 1$. This is a contradiction since $|J| = 2(s - 1)$. The proof is now complete.

Theorem 10 Let $n, s$ be positive integers with $s ≤ n$. Then the property "$\alpha(G) ≤ s$" is $(2n - 2s - 1)$-degree stable.

Proof. Suppose that $\alpha(G + ab) ≤ s$ while $\alpha(G) > s$. Then there must exist an independent set $W \cup \{a, b\} \subset V$ with $(s + 1) ≥ 3$ vertices. More precisely
$W \subseteq T$. Now $d(a) + d(b) \leq 2\gamma_{ab} = 2(n - \overline{\pi}_{ab})$. By the hypothesis $d(a) + d(b) + (\overline{\pi}_{ab} - \alpha_{ab}) \geq (2n - 2s - 1)$. It follows that $2(n - \overline{\pi}_{ab}) + (\overline{\pi}_{ab} - \alpha_{ab}) \geq (2n - 2s - 1)$, that is $\overline{\pi}_{ab} + \alpha_{ab} < 2(s + 1)$. On the other hand $\alpha_{ab} \geq |W \cup \{a, b\}| = s + 1$. Moreover $\overline{\pi}_{ab} \geq \alpha_{ab} \geq s + 1$. With this contradiction, Theorem 10 is proved. ■

**Theorem 11** Let $n, s$ be positive integers with $s \leq n - 2$. Then the property of being $s$-connected is $(n + s - \overline{\pi}_{ab})$-(degree stable).

**Proof.** Suppose that $G + ab$ is $s$-connected but $G$ is not. Then there exists a set $D$ of $(s - 1)$ vertices such that $a$ and $b$ belong to two distinct components of $G - D$. It follows in particular that $\lambda_{ab} < s$. By the hypothesis $d(a) + d(b) + (\overline{\pi}_{ab} - \alpha_{ab}) \geq (n + s - \overline{\pi}_{ab})$. As $d(a) + d(b) = \gamma_{ab} + \lambda_{ab}$ and $\overline{\pi}_{ab} + \gamma_{ab} = n$ we get $n - \overline{\pi}_{ab} + \lambda_{ab} + (\overline{\pi}_{ab} - \alpha_{ab}) \geq (n + s - \overline{\pi}_{ab})$, that is $\lambda_{ab} - \alpha_{ab} \geq s - \overline{\pi}_{ab}$. As $\lambda_{ab} < s$ we obtain $\alpha_{ab} \leq 1$, contradicting the fact that $\alpha_{ab} \geq 2$. This completes the proof. ■

Note that even if $(\overline{\pi}_{ab} - \alpha_{ab}) = 0$, this result improves Bondy-Chvátal’s result in [7].

**Theorem 12** Let $n, s$ be positive integers with $s \leq n - 2$. Then the property of being $s$-edge-connected is $(n + s - \overline{\pi}_{ab})$-(degree stable).

**Proof.** Suppose that $G + ab$ is $s$-edge-connected but $G$ is not. Then there exists a set $F$ of $(s - 1)$ edges such that $a$ and $b$ belong to two distinct components of $G - F$. It follows in particular that $\lambda_{ab} < s$. The remaining of the proof follows that of the preceding Theorem. ■

4 Corollaries

The following results can be easily derived as Corollaries. Let $G$ be a graph of order $n$, $S$ be a subset of vertices and $s \leq |S|$ be an integer.

Let $c(G)$ denote the circumference of $G$. The first Corollary follows easily from Theorem 2.

**Corollary 1** The property $c(G) \geq s$ is $n$-degree stable.

The graph $G$ is $S$-pancyclicable if, for every integer $s$, with $3 \leq s \leq n$, there exists a cycle $C$ in $G$ such that $|S \cap V(C)| = s$. As usual, $G$ is pancyclic if it contains cycles of all lengths from 3 to $n$.

**Corollary 2** The property "$G$ is $S$-pancyclicable" is $(n + s - 3)$ - degree stable with $3 \leq s \leq n$.

**Proof.** Let $R \subseteq S$ be a subset of $r$ vertices, $3 \leq r \leq s$, which is not contained in any cycle $C$ of $G$. This means that $G - (S \setminus R)$ is not hamiltonian, in other words $G$ is not $(s - r)$-hamiltonian. By Theorem 3, $d_G(a) + d_G(b) + (\overline{\pi}_{ab} - \alpha_{ab}) < n + s - r \leq n + s - 3$, a contradiction to the hypothesis. ■
Corollary 3 The property "G is pancyclic" is \((2n - 3)\) - degree stable.

Proof. By identifying \(S\) and \(V\) in the preceding Corollary, we are done. ■

Corollary 4 The property of being Hamiltonian-connected is \((n+1)\)-degree stable.

Proof. Follows from Theorem 6 with \(s = 2\) or Theorem 8 with \(F = \emptyset\). ■

The graph \(G\) is \(S\) - vertex Hamiltonian-connected if it remains Hamiltonian-connected if \(s\) vertices of \(S\) or less are removed. Using similar arguments as for Theorem 6 we get:

Corollary 5 The property "\(S\)- vertex Hamiltonian-connected" is \((n + s + 1)\)-degree stable.

Applying Theorem 7, we easily get:

Corollary 6 The property of being \(s\)-edge-hamiltonian is \((n + s)\)-degree stable.

Let \(\mu(G)\) be the number of paths that collectively contain the vertices of \(G\).

Corollary 7 The property \(\mu(G) \leq p, 1 \leq p \leq n\) is \((n - p)\)-degree stable.

Proof. Apply Theorem 2 for the graph \(G + pK_1\). ■

The graph \(G\) is called \(S\)-leaf-connected if it has a spanning tree whose leaves are the vertices of \(S\). Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected.

Corollary 8 Let \(S \subset V(G)\) with \(s\) vertices, \(2 \leq s \leq n\). Then the property "\(G\) is \(S\)-leaf-connected" is \((n + s - 1)\) - stable.

Proof. This is a particular case of Theorem 6. ■

4.1 Open Problem

We believe that the following must be true.

Problem 1 Let \(n, s\) be positive integers with \(2 \leq s < n\). Then the property of having an \(s\)-factor is \((n + 2s - 4)\)-degree stable.
5 A polynomial version of the $\alpha$-degree closure

To improve Bondy-Chvátal’s closure condition we have added $(\pi_{ab} - \alpha_{ab})$ to $\sigma_{ab}$ in order to define $P(k)$. This can be indeed a large number but $\alpha_{ab}$ is hard to compute. This motivates us to introduce some easy computable upper bounds of $\alpha_{ab}$ (or alternatively lower bounds of $(\pi_{ab} - \alpha_{ab})$).

The first lower bound is based on the matching number $\nu_{ab}$ of the graph $G_{ab}$ as it is well known that $\alpha(H) \leq |H| - \nu(H)$ holds for any graph. In particular we have $\alpha_{ab} \leq \pi_{ab} - \nu_{ab}$ if $H = G_{ab}$. It is worth noting that for any subgraph $H$ of $G$ we have $\nu(H) \leq \nu(G)$.

For the second lower bound, we introduce a new invariant of a graph based on the degree sequence of that graph.

**Definition 2** Let $H$ be any graph of order $n$ and $\theta$ be a nonnegative integer. Set $D_{\theta} := \{x \in V(H) \mid d_{x}(x) \geq \theta\}$. The adjusted maximum degree $\Delta^{\theta}(H)$ is the maximum integer $\theta$ such that $|D_{\theta}| \geq \theta$.

In fact, we are mainly interested by this invariant when applied to $G_{ab}$. Thus we have $\Delta^{\alpha}_{ab} := \max \{\theta \mid |D_{\theta}| \geq \theta\}$ where $D_{\theta} := \{x \in T \mid d_{T}(x) \geq \theta\}$. The next Proposition precises some properties of this new invariant.

**Proposition 2** The invariant $\Delta^{\alpha}(G)$ satisfies the following properties:

1. $\Delta^{\alpha}(G)$ does not necessarily correspond to a degree of some vertex of $G$,
2. $\Delta^{\alpha}(G)$ is well-defined and $0 \leq \delta(G) \leq \Delta^{\alpha}(G) \leq \Delta(G) \leq n - 1$,
3. the invariants $\Delta^{\alpha}(G)$ and $\nu(G)$ are incomparable,
4. for any subgraph $H$ of $G$ with $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ we have $\Delta^{\alpha}(H) \geq \Delta^{\alpha}(G) - |V(G)\setminus V(H)|$. Similarly $\nu(H) \geq \nu(G) - |V(G)\setminus V(H)|$.
5. $\Delta^{\alpha}(G) = \{|i| + d_{i} > n\}$ where $d_{1} \leq d_{2} \leq \ldots \leq d_{n}$ is the degree sequence of $G$.
6. $\alpha(G) \leq n - \max \{\nu, \Delta^{\alpha}\}$ and hence $\alpha_{ab}(G) \leq \pi_{ab}(G) - \max \{\nu_{ab}, \Delta^{\alpha}_{ab}\}$.

**Proof.**
1. For instance, if the degree sequence is $(2, 2, 2, 4, 4, 4)$ then $\Delta^{\alpha} = 3$ since $|D_{3}| \geq 3$, while $|D_{4}| < 4$.
2. Obvious.
3. For instance, if $G = pC_{k}$, $k \geq 3$, $p \geq 1$ then $\delta = \Delta^{\alpha} = \Delta = 2$ and $\nu(G) = p \left[\frac{k}{2}\right]$. Similarly if $G = K_{2}$ then $\delta = \Delta^{\alpha} = \Delta = n - 1$ and $\nu(G) = \left[\frac{n}{2}\right]$.
4. The inequalities are obvious if $V(H) = V(G)$. Otherwise, use a simple induction on $|V(H)|$.
5. Suppose that $G$ is not trivial since otherwise $\Delta^{\alpha}(G) = 0$. Choose $p \in [1, n]$ so that $p := \min \{|i| + d_{i} > n\}$ and set $\theta := \{|i| + d_{i} > n\}$. Clearly $\theta = n + 1 - p$. We claim that $d_{i} \geq \theta$ whenever $i \geq p$. Otherwise suppose $d_{p} < \theta$. Then $\theta = n + 1 - p > d_{p}$, that is $p + d_{p} \leq n$. This contradicts the definition of $p := \min \{|i| + d_{i} > n\}$.
6. Suppose first \( \max \{ \nu, \Delta^o \} = \Delta^o(G) \). Let \( H \) be any component of \( G \) and consider a maximum independent set \( S := \{ x_1, \ldots, x_{\alpha(H)} \} \) of \( H \). We label the vertices of \( H \) so that \( d(x_1) \leq d(x_2) \leq \cdots \leq d(x_{\alpha(H)}) \). Clearly \( d(x_{\alpha(H)}) + |S| \leq |H| \), that is
\[
d(x_{\alpha(H)}) + \alpha(H) \leq |H|.
\]
So \( d_H(x_i) + i \leq n \) must be true for all \( i \leq \alpha(H) \). If, for some \( j > \alpha(H) \) we have \( d_H(x_j) + j > |H| \), then necessarily \( x_j \in V(H) \setminus S \). Therefore \( \{x_j \mid j > \alpha(H) \} \subseteq V(H) \setminus S \), that is \( \Delta^o(H) = |\{x_j \mid j > \alpha(H) \}| \leq |H| - \alpha(H) \) or \( \alpha(H) \leq |H| - \Delta^o(H) \). Applied to the graph \( G_{ab} \), this inequality becomes \( \alpha_{ab} + \Delta^o_{ab} \leq \pi_{ab} \).
It is well known that \( \alpha(G) + \nu \leq n \) holds for any graph. Therefore \( \alpha(G) \leq n - \max \{ \nu, \Delta^o \} \) holds again if \( \max \{ \nu, \Delta^o \} = \nu \).

We note that the proof of 6 of Proposition 2 suggests an interesting result in itself, that is:

**Proposition 3** Let \( G \) with degree sequence \( d_1 \leq \ldots \leq d_n \). Then \( d_n + \alpha \leq n \).

Moreover Proposition 2 suggests an alternative condition for \( P(k) \), namely:

**Definition 3** Let \( P \) be a property defined for all graphs \( G \) of order \( n \) and let \( k \) be an integer. Let \( a, b \) be two nonadjacent vertices satisfying the condition
\[
P^*(k) : \quad \pi_{ab} \leq \lambda_{ab} + \max(\nu_{ab}, \Delta^0_{ab}) + (n - k) \Leftrightarrow \sigma_{ab} + \max(\nu_{ab}, \Delta^0_{ab}) \geq k. \quad (**)
\]
Then \( P \) is \( k \)-alpha degree stable if whenever \( G + ab \) has property \( P \) and \( P^*(k) \) holds then \( G \) itself has property \( P \). We simply denote by \( dC^*_k(G) \) the associated \( (\alpha \text{-degree closure}) \).

The graph \( dC^*_k(G) \) is then obtained from \( G \) by recursively joining pairs of nonadjacent vertices \( a, b \) for which \( (** \text{ holds}) \) until no such pair remains.

Unlike \( dC_k(G) \), the closure graph \( dC^*_k(G) \) can be constructed in polynomial time. Obviously, the main results given in section 3 remain true with the new definition of \( P(k) \).

Also with the following Proposition we obtain a surprising result involving the main closure condition of Broersma and Schiermeyer [8].

**Proposition 4** Let \( (a, b) \) be a pair of nonadjacent vertices of a graph \( G \). Suppose \( T \neq \emptyset \) and let \( d_T(x) \) denote the degree of \( x \in T \) with respect to \( G \) \( \Gamma_{ab} = |N(a) \cup N(b) \cup N(x)| \). Then
\[
|\{x \in T \mid \gamma_{abx} \geq n - \lambda_{ab} \}| \geq \pi_{ab} - \lambda_{ab} \Rightarrow \quad (3)
\]
\[
\pi_{ab} \leq \lambda_{ab} + \Delta^0_{ab} \Rightarrow \quad (4)
\]
\[
\alpha_{ab} \leq \lambda_{ab}. \quad (5)
\]

Note that \( \pi_{ab} \leq \lambda_{ab} + \Delta^0_{ab} \) is equivalent to \( \sigma_{ab} + \Delta^0_{ab} \geq n \).
Proof. We first note that $\gamma_{abz} \geq n - \lambda_{ab} \iff d_T(x) \geq \pi_{ab} - \lambda_{ab}$ since $\pi_{ab} = n - \gamma_{ab}$ and $\gamma_{abz} = \gamma_{ab} + d_T(x)$. Therefore $|\{x \in T | \gamma_{abz} \geq n - \lambda_{ab}\}| \geq |\pi_{ab} - \lambda_{ab}| \geq \pi_{ab} - \lambda_{ab}$. By the definition, $\Delta_{ab}^o \geq |\{x \in T | d_T(x) \geq \pi_{ab} - \lambda_{ab}\}|$ and hence the Broersma-Schiermeyer’s inequality becomes $\Delta_{ab}^o \geq \pi_{ab} - \lambda_{ab}$. This proves (4). By 6. of Proposition 2 $\alpha_{ab} + \Delta_{ab}^o \leq \pi_{ab}$. This completes the proof of the above Proposition. ■

In other words, the main part of Theorem 2.1 [8] can be restated as follows:

**Theorem 13** Let $u$ and $v$ be two nonadjacent vertices of a graph $G$ of order $n \geq 3$. If $d(u) + d(v) + \Delta_{uv} \geq n$ then $G$ is hamiltonian if and only if $G + uv$ is hamiltonian.

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**References**


