“On nontrivial solution for a quasilinear elliptic system involving variable exponents via a sub-supersolution method”

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Abstract

In this paper, we present some existence results concerning in a typical \((p(x), q(x))\)-gradient elliptic system with changing sign. We construct a pair of sub-super solutions. The existence of a positive and bounded solution involves.

Key Words: Sub-supersolutions, \(p(x)\)-Laplacian, minimization.

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1 Introduction

During the recent decade, a numerous of papers have been devoted to the study of problems with variable exponent. These problems involving the \(p(x)\)-Laplacian operator arise in the modeling of electrorheological fluids (see [18]) and image restorations among other problems in physics and engineering. \(p(x)\)-laplace equations also arise from elastic mechanics [22].

Also, these equations or systems governed by \(p(x)\)-Laplacian rise many mathematical difficulty respect with the classical \(p\)-Laplacian operator. It well known that \(p(x)\)-Laplacian operator is more complicated than the classical \(p\)-Laplacian operator. Thank to the recent development of the theory of variable exponent Lebesgue and Sobolev spaces, an extensive literature has appeared on solving \(p(x)\)-Laplacian equation. For problem involving systems of equations like

\[
\begin{aligned}
-\Delta_{p(x)}u &= B_1(x, u, v) \\
-\Delta_{q(x)}v &= B_2(x, u, v).
\end{aligned}
\]  

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mixed with Dirichlet boundary conditions have been investigated. For instance, weak solutions of such systems have been obtained more recently. We can cite Diening and Ruzicka. In the case \( p(\cdot) = q(\cdot) \), Zhang [20] establishes results by considering \( B_1(x, u, v) = f(v) \) and \( B_2(x, u, v) = g(u) \). Other results can also be consulted in [21].

The case \( p(\cdot) \neq q(\cdot) \) has been investigated by [1] when \( \Omega \) is a ball in \( \mathbb{R}^N \). These authors prove the existence of a positive weak solution by considering \( p \) of class \( C^1(\Omega) \), \( B_1(x, u, v) = \lambda (h(x)a(u) + f(v)) \) and \( B_2(x, u, v) = \lambda (h(x)b(v) + f(u)) \).

The \((p(x), q(x))\)-gradient elliptic system has been considered by El Hamidi in [3]. More precisely, when \( B_1(x, u, v) = \frac{\partial F}{\partial u}(x, u, v) \) and \( B_2(x, u, v) = \frac{\partial F}{\partial v}(x, u, v) \), \( F \) has polynomial growth, the author shows there exists at least one weak solution in \( W_0^{1,p(x)}(\Omega) \). When \( F \) is even in the second and the third variable, the author shows that the problem admits an infinitely many weak solutions in \( W_0^{1,p(x)}(\Omega) \). Others variants of the structure like (1.1) have been considered. One can cited X. Xu and Y. An [19], Ogras, Mashiyev, Avci and Yucedag, [17], Liu and Shi [14].

Recently, the system (1.1) has been considered with \((p(x), q(x))\)-gradient structure i.e \( B_1(x, u, v) = \frac{\partial F}{\partial u}(x, u, v) \) and \( B_2(x, u, v) = \frac{\partial F}{\partial v}(x, u, v) \), avec \( F(x, u, v) = c(x)|u|^{\alpha+1}v^{\beta+1} \). Particularly, nonexistence and existence results have been obtained. For instance, for \( \alpha, \beta, p^-, q^- \) and \( N \) such that
\[
\frac{(\alpha + 1) N - p^-}{Np^-} + (\beta + 1) \frac{N - q^-}{Nq^-} < 1 \text{ and } \frac{\alpha + 1}{p^-} + \frac{\beta + 1}{q^-} - 1 > 0,
\]
an existence result of positive solution have been established by using a fibering method.

In the present paper, we treat about the system (1.1) with the structure
\[
F(x, u, v) = c(x)|u|^{\alpha+1}v^{\beta+1}.
\]
In this case, (1.1) becomes
\[
\begin{align*}
-\Delta_{p(x)} u &= c(x)u|u|^{\alpha-1}|v|^\beta+1 & \text{in } \Omega \\
-\Delta_{q(x)} u &= c(x)|u|^{\alpha+1}v^{\beta-1} & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega.
\end{align*}
\]

By using a sub-super solution method, we deal with the case \( \frac{\alpha + 1}{p^-} + \frac{\beta + 1}{q^-} - 1 < 0 \). After constructing respectively a super-solution \((u^0, v^0)\) and a sub-solution \((u_0, v_0)\), we show that the problem admits at least a positive solution \((u^*, v^*)\) such that \( u_0 \leq u^* \leq u^0 \) and \( v_0 \leq v^* \leq v^0 \).

The purpose of the problem is done above. Let us announce the organization convened in this paper:

- The first section is devoted to recall the main material needed for establishing of our results.
In the second section, we construct a pair of super-solution and sub-solution for the system.

In the third section, we associate to a truncated system which we establish the existence of a solution via a minimizing approach.

In the last section, we establish that the solution is located in $[u_0, u^0] \times [v_0, v^0]$.

2 Preliminaries, notation

2.1 Some results on Lebesgue and Sobolev spaces

In this section, and throughout the study, we recall some definitions and properties on the generalized Lebesgue space $L^p(x)(\Omega)$ and generalized Sobolev spaces $W^{1,p}(x)(\Omega)$. $\Omega \subset \mathbb{R}^N$ is an open set. For more details, the reader can consult for instance [2, 5, 7, 8, 11, 12, 13, 15, 16].

The generalized Lebesgue space $L^p(x)(\Omega)$ consists in all measurable functions $u$ defined on $\Omega$ for which the $p(x)$-modular

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx$$

is finite. The Luxemburg norm on this space is defined as:

$$\|u\|_{L^p(x)(\Omega)} = \inf \{ \lambda > 0; \rho_{p(\cdot)}(u) = \int_{\Omega} \frac{|u(x)|^{p(x)}}{\lambda} \, dx \leq 1 \}.$$

Equipped with this norm, $L^p(x)(\Omega)$ is a Banach space.

If $p(x)$ is constant, $L^p(x)(\Omega)$ is reduced to the standard Lebesgue space.

For given $p \in L^\infty(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

The following results show the close relation between the convex modular $\rho_{p(\cdot)}$ and the norm $\|\cdot\|_{L^p(\cdot)(\Omega)}$.

Let us recall main results on generalized Lebesgue spaces. We start by

**Proposition 2.1** Let $p \in L^\infty(\Omega)$.

1. If $u \in L^p(\cdot)(\Omega)$ then $\|u\|_{L^p(\cdot)(\Omega)} = a \iff \varphi\left(\frac{u}{a}\right) = 1$

2. $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (=1, >1) \iff \varphi_{p(\cdot)}(u) < 1 (=1, >1)$
3. If \( \|u\|_{L^p(\cdot)} > 1 \) then \( \|u\|_{L^{p_+}(\cdot)}^{p_+} \leq q_p(\cdot)(u) \leq \|u\|_{L^{p_+}(\cdot)}^{p_-} \)

4. If \( \|u\|_{L^p(\cdot)} < 1 \) then \( \|u\|_{L^{p_+}(\cdot)}^{p_+} \leq q_p(\cdot)(u) \leq \|u\|_{L^{p_+}(\cdot)}^{p_-} \)

**Proposition 2.2** \([13, 5]\) Let \( p \in L^\infty_+ (\Omega) \), \( (u_n) \subset L^{p(\cdot)}(\Omega) \) and \( u \in L^{p(\cdot)}(\Omega) \).

The following assertions are equivalent:

1. \( \lim_{n \to +\infty} \|u - u_n\|_{L^{p(\cdot)}} = 0 \)
2. \( \lim_{n \to +\infty} q_{p(\cdot)}(u - u_n) = 0 \).

**Theorem 2.1** (see\([5, 6, 16]\)). Consider \( p, q, r \in L^\infty_+ (\Omega) \), \( u \in L^{q(\cdot)}(\Omega) \) et \( v \in L^{r(\cdot)}(\Omega) \) such that:

\[
\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)} \quad e. \ a \ in \ \Omega
\]

then

\[
\|uv\|_{L^{r(\cdot)}(\Omega)} \leq \left[ \frac{1}{(p/r)_-} + \frac{1}{(q/r)_-} \right] \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{r(\cdot)}(\Omega)}
\]

for all \( u \in L^{q(\cdot)}(\Omega) \), \( v \in L^{r(\cdot)}(\Omega) \). It is immediate to make this remark

**Remark 1** Let \( p \in L^\infty_+ (\Omega) \) and let \( p' : \Omega \to [1, +\infty[ \) be the conjugate function of \( p \).

There is a constant \( C_p > 0 \) such that:

\[
\int_\Omega |uv| \leq C_p \|u\|_{L^{p(\cdot)}} \|v\|_{L^{p'(\cdot)}}
\]

for all \( u \in L^{p(\cdot)}(\Omega) \), \( v \in L^{p'(\cdot)}(\Omega) \).

We also have the following imbedding theorem and we refer the reader to Kovacik and Rokosnik\([13]\), Fan and Zhao\([5]\)

**Proposition 2.3** \([13]\) Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and let \( p, q \in L^\infty_+ (\Omega) \).

If \( p(x) \leq q(x) \ a.e \ in \ \Omega \), then \( L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \).

Now, we recall main results about generalized Sobolev space. For any \( p \in L^\infty_+ (\Omega) \) and \( m \in \mathbb{N}^* \), we define

\[
W^{m,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega) \text{pour tout } |\alpha| \leq m \},
\]

\[
\|u\|_{m,p(\cdot)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}
\]

The pair \((W^{m,p(\cdot)}(\Omega), \|\cdot\|_{m,p(\cdot)})\) is a separable Banach space (reflexive if \( p_- > 1 \)) which is called generalized Sobolev space (also known as Sobolev space with variable exponent). We will denote by \( W^{1,p(\cdot)}_0(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in \( W^{m,p(\cdot)}(\Omega) \).
Proposition 2.4 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $p, q \in L_+^{\infty}(\Omega)$. If

$$p(x) \leq q(x) \ a.e \ in \ \Omega,$$

then

$$W^{1,q(.)}(\Omega) \hookrightarrow W^{1,p(.)}(\Omega).$$

Based on the $p(.) - \text{Capacity}$ notion, we also have:

Proposition 2.5 [10] Let $1 < q^-, p^+ < +\infty$ and $p(x) \geq q(x)$ for almost every $x \in \mathbb{R}^N$. Assume that $\Omega \in \mathbb{R}^N$ is a bounded open set. Then

$$W_0^{1,p(.)}(\Omega) \hookrightarrow W_0^{1,pl(.)}(\Omega).$$

Moreover, the norm of the embedding operator does not exceed $1 + |\Omega|.$

Definition 2.1 We say that a function $p : A \to \mathbb{R}$ is ln-Hölder continuous on $A$ provided that there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\ln |x - y|}$$

for all $x, y \in A$, $|x - y| \leq \frac{1}{2}.$

The following density result holds.

Theorem 2.2 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and $p \in L_+^{\infty}$. If $p$ is ln- Hölder continuous on $\Omega$, then $C^\infty(\Omega)$ is dense in $W^{1,p(.)}(\Omega)$.

Theorem 2.3 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $p \in C(\Omega)$ be a function which satisfies $p- > 1$.

Define the Sobolev conjugate exponent $p^* : \Omega \to \mathbb{R}$ of $p$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

then the imbedding $W^{m,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and holds for every function $q \in C(\Omega)$ which satisfies $1 < q(x) < p^*(x)$ for all $x \in \Omega.$
2.2 Main results

The solvability of problems quasilinear governed by $p(x)$-laplacian can be studied by several approaches like variational method. Among them, we cite the sub-supersolution method. The reader interested by the applications to semilinear and quasilinear elliptic problems governed by the $p(x)$-laplacian can refer to XL. Fan [4] and the references therein, X. Han and G. Dai [9] for a version applied to Kirchhoff type equations. The sub-supersolution principle for $p(x)$-laplacian is based on the properties of $p(x)$-laplace and also from the results obtained in [4], we observe that the general principle of sub-supersolution method for the problem with variable exponent is the same type as in the constant exponent case. An essential prerequisite for the sub-supersolution method is to find a subsolution $u_0$ and a supersolution $v_0$ such that $u_0 \leq v_0$.

3 Weak solution, weak super-solution and weak sub-solution for (1.2)

The notation remain the same as above, we set: $B_1(x, u, v) = c(x)|u|^{\alpha-1}|v|^{\beta+1}$ and $B_2(x, u, v) = c(x)|u|^{\alpha+1}|v|^{\beta-1}$.

3.1 Definition

Definition 3.1 $(u^*, v^*) \in W^{1,p(x),q(x)}$ is a weak solution of (1.2) if

$$
\int_\Omega |\nabla u^*|^{p(x)-2} \nabla u^* \nabla w_1 dx + \int_\Omega |\nabla v^*|^{q(x)-2} \nabla v^* \nabla w_2 dx
- \int_\Omega B_1(x, u^*, v^*) w_1 dx - \int_\Omega B_2(x, u^*, v^*) w_2 dx = 0
$$

for any $(w_1, w_2) \in W^{1,p(x),q(x)}$.

Definition 3.2 $[(u_0, v_0), (u_0^0, v_0^0)] \in W^{1,p(x)}_0 (\Omega) \times W^{1,q(x)}_0 (\Omega)$ is a weak supersolution of the Dirichlet problem associated to the system (1.2), if the following condition holds:

$$
\begin{align*}
\Delta_{p(x)} u_0^0 + B_1(x, u_0^0, v_0) &\leq 0 \leq \Delta_{p(x)} u_0 + B_1(x, u_0, v) & \text{in } \Omega, \forall v \in [v_0, v_0^0] \\
\Delta_{q(x)} v_0^0 + B_2(x, u, v_0^0) &\leq 0 \leq \Delta_{p(x)} v_0 + B_2(x, u, v_0) & \text{in } \Omega, \forall u \in [u_0, u_0^0] \\
u_0 \leq u_0^0, \quad v_0 \leq v_0^0 & \quad \text{in } \Omega, \\
u_0^0 \leq u_0, \quad v_0 \leq 0 \leq v_0^0 & \quad \text{on } \partial\Omega.
\end{align*}
$$

Before starting the next section, we need some lemmas.
Lemma 3.1 Assume that $p = p(\|x\|)$ is a radial function. Then, the functionals $\varphi$ and $\phi$ defined on $[0, R^0]$ as follow

$$
\varphi(r) = a_p \left[ (R^0 - \delta r)^{\mu^0 - 1} + 1 \right], \quad \phi(r) = a_q \left[ (R^0 - \delta r)^{\mu^0 - 1} + 1 \right] \tag{3.1}
$$

obey to

$$
\Delta_{p(x)} \varphi(r) + c(x) \varphi |^{\alpha - 1} \phi |^{\beta + 1} \leq 0 \tag{3.2}
$$

where

$$
\inf (a_p, a_q) > \left\{ \frac{R^0}{\delta (\mu^0 - 1)} \right\}, \quad 0 < \delta < 1 \text{ and } 1 < \mu^0 < 2.
$$

**Proof.** For $r$ such that $0 \leq r \leq R^0$, we take

$$
\varphi(r) = \tilde{a} \left[ (R^0 - \delta r)^{\mu^0 - 1} + 1 \right].
$$

From the hypothesis on $\tilde{a}$, it is clear that $\varphi$ is a positive $C^2$ function on $[0, R^0]$. Moreover, we have

$$
\Delta_{p(x)} \varphi(r) = \left[ p'(r) \ln(|\varphi'(r)|) + \frac{N - 1}{r} \right] |\varphi'(r)|^{p(r)-2} \varphi'(r) + (p(r)-1) |\varphi'(r)|^{p(r)-2} \varphi''(r).
$$

So on, by computation, for any $r$ such that $0 \leq r \leq R^0$, we get

$$
\varphi'(r) = -\delta (\mu^0 - 1) a_p (R^0 - \delta r)^{\mu^0 - 2},
$$

and

$$
\varphi''(r) = \delta^2 (\mu^0 - 1) (\mu^0 - 2) a_p (R^0 - \delta r)^{\mu^0 - 3}.
$$

We deduce successively:

$$
|\varphi'(r)|^{p(r)-2} = \delta^{p(r)-2} (\mu^0 - 1)^{p(r)-2} a_p^{p(r)-2} (R^0 - \delta r)^{(\mu^0-2)(p(r)-2)},
$$

$$
|\varphi'(r)|^{p(r)-2} \varphi'(r) = -\delta^{p(r)-1} (\mu^0 - 1)^{(p(r)-1) a_p^{p(r)-1} (R^0 - \delta r)^{(\mu^0-2)(p(r)-1)},}
$$

$$
\ln(|\varphi'(r)|) = \ln \left( \delta (\mu^0 - 1) a_p \left\{ (R^0)^{\mu^0 - 2} \right\} + (\mu^0 - 2) \ln \left( 1 - \frac{\delta R}{R} \right) \right).
$$

In the right hand of $\Delta_{p(x)} \varphi(r)$, we estimate each of both terms:

1. Estimation of the term $(p(r) - 1) |\varphi'(r)|^{p(r)-2} \varphi''(r)$:

   We have

   $$
   |\varphi'(r)|^{p(r)-2} \varphi''(r) = (\mu^0 - 2) \delta^{p(r)} (\mu^0 - 1)^{p(r)-1} a_p^{p(r)-1} (R^0 - \delta r)^{p(r)-2}(p(r)-1).
   $$

7
So, thank the assumption on \( \mu^0 \), it derives
\[
|\varphi'(r)|^{p(r)-2} \varphi''(r) \leq (\mu^0 - 2) \delta \varphi'(r) (\mu^0 - 1)^{p(r)-1} \tilde{a}^{p(r)-1} R^{(\mu^0-2)(p(r)-1)-1}.
\]
Because \( 0 < \delta < 1 \), it follows that
\[
(p(r) - 1) |\varphi'(r)|^{p(r)-2} \varphi''(r) \leq -K_1 \tilde{a}^{p(r)-1}.
\]
where \( K_1 = (p^r - 1)(\mu^0 - 2)\delta \tilde{a}^{p(r)-1} R^{(\mu^0-2)(p(r)-1)-1} \).

2. Estimation of the term
\[
\left| p'(r) \ln(|\varphi'(r)|) + \frac{N - 1}{r} \right| |\varphi'(r)|^{p(r)-2} \varphi'(r).
\]
Before, let us define on \([0, R^0]\) the function \( l \) as follows
\[
l(r) = -p'(r) \left[ \ln \left( \delta (\mu^0 - 1) \tilde{a} \{ R^0 \} R^{\mu^0-2} \right) \right] + (\mu^0 - 2) \ln \left( 1 - \frac{\delta r}{R^0} \right) |\delta \tilde{a}|^{p(r)-1} (\mu^0 - 1)^{p(r)-1} (R^0 - \delta r) (\mu^0-2)(p(r)-1).
\]
From the hypothesis \( \tilde{a} > \frac{\{ R^0 \}^{2-\mu^0}}{\delta (\mu^0 - 1)} \), we have successively \( \delta (\mu^0 - 1) \tilde{a} \{ R^0 \} R^{\mu^0-2} > 1 \) and so, \( \ln \left( \delta (\mu^0 - 1) \tilde{a} R^0 R^{\mu^0-2} \right) > 0 \). Moreover, remember us that \( 1 < \mu < 2 \), the following estimate on \( l(r) \) holds
\[
l(r) \leq - \inf_{0 \leq r \leq R^0} p'(r) \ln \left[ \delta (\mu^0 - 1) \tilde{a} \{ R^0 \} R^{\mu^0-2} \right] |\delta \tilde{a}|^{p(r)-1} (\mu^0 - 1)^{p(r)-1} \{ R^0 \} (\mu^0-2)(p(r)-1) \times \sup_{0 \leq r \leq R^0} p'(r) (\mu^0 - 2) \inf_{0 \leq r \leq R^0} \left[ \left( 1 - \frac{\delta r}{R^0} \right) R^{\mu^0-2}(p(r)-1) \right] \times \\
\ln \left( 1 - \frac{\delta r}{R^0} \right) |\delta \tilde{a}|^{p(r)-1} (\mu^0 - 1)^{p(r)-1} \{ R^0 \} (\mu^0-2)(p(r)-1).
\]
We set
\[
K_2 = - \inf_{0 \leq r \leq R^0} p'(r) \ln \left[ \delta (\mu^0 - 1) \tilde{a} \{ R^0 \} R^{\mu^0-2} \right] |\delta \tilde{a}|^{p(r)-1} (\mu^0 - 1)^{p(r)-1} \{ R^0 \} (\mu^0-2)(p(r)-1) \times \\
+ \sup_{0 \leq r \leq R^0} p'(r) (\mu^0 - 2) \inf_{0 \leq r \leq R^0} \left[ \left( 1 - \frac{\delta r}{R^0} \right) R^{\mu^0-2}(p(r)-1) \right] \times \\
\ln \left( 1 - \frac{\delta r}{R^0} \right) |\delta \tilde{a}|^{p(r)-1} (\mu^0 - 1)^{p(r)-1} \{ R^0 \} (\mu^0-2)(p(r)-1).
\]
We conclude that
\[
p'(r) \ln(|\varphi'(r)|) |\varphi'(r)|^{p(r)-2} \varphi'(r) \leq -K_2 \tilde{a}^{p(r)-1}.
\]
On other side,
\[
\frac{N-1}{r} |\varphi'(r)|^{p(r)-2} \varphi'(r) = -\frac{N-1}{r} \left[ (\mu - 1) \delta \tilde{\alpha}(R^0 - \tilde{r})^{\mu^0 - 2} \right]^{(p(r)-1)} \\
\leq -\frac{N-1}{R^0} \left[ (\mu^0 - 1) \delta \{R^0\}^{\mu^0 - 2} \right]^{(p(r)-1)} \tilde{\alpha}^{(p(r)-1)}.
\]

So, setting \( K_3 = \frac{N-1}{R^0} \left[ (\mu^0 - 1) \delta \{R^0\}^{\mu^0 - 2} \right]^{(p(r)-1)} \), we obtain
\[
\frac{N-1}{r} |\varphi'(r)|^{p(r)-2} \varphi'(r) \leq -K_3 \tilde{\alpha}^{(p(r)-1)}. \quad (3.5)
\]

Using again the definition (3.1), it follows:
\[
\left[ p'(r) \ln(|\varphi'(r)|) + \frac{N-1}{r} \right] |\varphi'(r)|^{p(r)-2} \varphi'(r) \\
= l(r) - \frac{N-1}{r} (\mu^0 - 1)^{(p(r)-1)} \tilde{\alpha}^{(p(r)-1)} (R^0 - \delta r)^{(\mu^0 - 2)(p(r)-1)}.
\]

Thanks to (3.3), (3.4) and (3.5) and denoting as \( K = K_1 + K_2 + K_3 \), thus, for any \( r \) such that \( 0 \leq r \leq R \), we obtain:
\[
\Delta_{p(x)} \varphi(r) \leq -K \tilde{\alpha}^{p(r)-1}.
\]

Before continuing, throughout the rest of the paper, let us assume that there is \( R_0 > 0 \) for which \( B(0; R_0) \subset \Omega \). We also distinguish the cases \( c(x) \geq 0 \) and \( c(x) < 0 \).

### 3.2 \( c(x) \geq 0 \)

Let us do the following hypotheses:

**Lemma 3.2** Assume that

1. \( R_0 = \min \left( \text{diam}(\Omega), \frac{(p^+ - 1)\mu^0}{\sup_{\Omega} |p'(r)|} \right) \).
   \[
   (3.6)
   \]

2. There exists a nonnegative constant \( c_1 \) such that \( \forall x \; c(x) > c_1 > 0 \),

3. For any \( r \geq R_0, \; p(r) > 2 \).
For any $s$ real, let $\psi_s$ be the radial function defined on $\Omega$ as follow:

$$
\psi_s(r) = \begin{cases} 
-\theta_1^{1/p^-(r^\gamma_p - B_s)} & \text{if } 0 \leq r \leq \frac{NR_0}{N+1}, \\
C_s(R_0 - r)^{\mu_s} & \text{if } \frac{NR_0}{N+1} \leq r \leq R_0 \\
0 & \text{if } R_0 \leq r.
\end{cases}
$$

(3.7)

Then, the pair $(\psi_p, \psi_q)$ is such that

$$
\Delta_{p(x)} \psi_p(x) + c(x)\psi_p|^{\alpha-1}\psi_q|^{\beta+1} \geq 0
$$

where

$$
B_p > \frac{\|c\|_{\infty}}{c_1} \left( \frac{NR_0}{N+1} \right)^{\gamma_p}, \quad \gamma_p > \frac{p^-}{p^--1}, \quad \gamma_q > \frac{q^-}{q^--1},
$$

$$
\theta_p \leq \min \left( \frac{1}{\gamma_p} \left( \frac{N+1}{NR_0} \right)^{\gamma_p-1}, \\
\left[ \inf_{0 \leq r \leq \frac{NR_0}{N+1}} \left| r^{\gamma_p} - B \right|^{\alpha-1} \left| r^{\gamma_q} - B \right|^{\beta+1} \left( -\|c\|_{\infty} \left( \frac{NR_0}{N+1} \right)^{\gamma_p} + Bc_1 \right) \right]^{-\frac{p^- - 1}{p^-}} \left( \frac{1}{\frac{p^- - 1}{p^-} + \frac{\beta+1}{q^-}} \right) \right),
$$

(3.8)

$$
0 < C_s < \frac{1}{\mu_p R_0^{\mu_p-1}}.
$$

Before starting the proof of this lemma, let us do a remark.

**Remark 2** If $\Omega$ is the ball $B(0, R_0)$, the hypothesis "There exists $R_0 > 0$, such that for any $r \geq R_0$, $s > 2$. becomes unnecessary. Consequently, we reduce the function $\psi_s$ as follow

$$
\psi_s(r) = \begin{cases} 
-\theta_1^{1/p^-}(r^{\gamma_s} - B_s) & \text{if } 0 \leq r \leq \frac{NR_0}{N+1}, \\
C_s(R_0 - r)^{\mu_s} & \text{if } \frac{NR_0}{N+1} \leq r \leq R_0.
\end{cases}
$$

So, in this case, we only suppose $p(r) > 1$.

**Proof.** Proof of Lemma 4.1. To ease the reading, we deal with the function $\psi_p$. We recall that

$$
\Delta_{p(x)} \psi_p(r) = \left[ p'(r) \ln(|\psi'_p(r)|) + \frac{N-1}{r} \right] |\psi'_p(r)|^{p(r)-2} \psi'_p(r) + (p(r)-1) |\psi'_p(r)|^{p(r)-2} \psi''_p(r).
$$
1. The case \(0 \leq r \leq \frac{NR_0}{N + 1}\).

We fix \(s = p\). Since the definition, \(\psi_p(r) = \theta_p^{1/p} (r^{\gamma_p} - B_p)\), by calculation, it is clear to obtain successively:

\[
p'(r) \ln \left(\left|\psi'_p(r)\right|^p\right) - \psi'_p(r) = -p'(r) \theta_p^{1/p} \gamma_p^{r^{\gamma_p-1}} \ln \theta_p^{1/p} \gamma_p^{r^{\gamma_p-1}} \\
\geq -p'(r) \theta_p^{1/p} \gamma_p^{r^{\gamma_p-1}} \ln \theta_p^{1/p} \gamma_p^{\left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1}} ,
\]

\[
\frac{N - 1}{r} \left|\psi'_p(r)\right|^p \psi'_p(r) = -\frac{N - 1}{r} \theta_p^{1/p} \gamma_p^{r^{\gamma_p-1}} \left|\psi'_p(r)\right|^p \\
\geq -(N - 1) \theta_p^{1/p} \gamma_p \left|\psi'_p(r)\right|^p \left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1} (r^{\gamma_p-1})^{p(r-1)-1}
\]

and

\[
(p(r) - 1) \left|\varphi'(r)\right|^p \varphi''(r) = -(p(r) - 1) \theta_p^{1/p} \gamma_p \left|\varphi'(r)\right|^p \\
\geq -(p(r) - 1) \theta_p^{1/p} \gamma_p \left|\varphi'(r)\right|^p \left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1} (r^{\gamma_p-1})^{p(r-1)-1} \\
\geq -p^+ \theta_p^{1/p} \gamma_p \left|\varphi'(r)\right|^p \left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1} (r^{\gamma_p-1})^{p(r-1)-1} \\
\geq -p^+ \theta_p^{1/p} \gamma_p \left|\varphi'(r)\right|^p \left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1} (r^{\gamma_p-1})^{p(r-1)-1} .
\]

To continue, from the hypothesis on \(\theta_p\), we remark that the term \(\ln \theta_p^{1/p} \gamma_p \left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1}\) is negative. Then, it derives that the term \(-\theta_p^{1/p} \gamma_p^{r^{\gamma_p-1}} \ln \theta_p^{1/p} \gamma_p^{\left(\frac{NR_0}{N + 1}\right)^{\gamma_p-1}}\) is also positive. That means, we get \(p'(r) \left|\psi'_p(r)\right|^p \psi'_p(r) \ln \left|\psi'_p(r)\right| > 0\).

So, taking account of the hypotheses on \(\gamma_p\), it derives that for \(r\) such that \(0 \leq r \leq \frac{NR_0}{N + 1}\), the following estimates holds:

\[
\Delta_p(x) \psi_p \geq - \left(N + p^+\right) \theta_p^{-} \gamma_p \left|\psi'_p(r)\right|^p \left(\frac{NR}{N + 1}\right)^{\gamma_p-1} (r^{\gamma_p-1})^{p(r-1)-1} .
\]
Moreover, it follows that
\[
\Delta_p(x) \psi_p(\|x\|) \geq (\frac{\alpha}{\alpha + 2 + \epsilon}) \left\{ -\frac{v - 1}{\alpha + 2 + \epsilon} \right\} (N + p^+) \gamma p^{-1} \left( \frac{NR}{N + 1} \right)^{(\gamma)(p - 1) - 1} + \mathop{\inf}_{0 \leq r \leq \frac{NR}{N + 1}} \left| r^{\gamma} - B \right|^{\alpha - 1} \left| r^{\gamma} - B \right|^{\beta + 1} \left( -\|c\|_{\infty} \left( \frac{NR}{N + 1} \right)^{\gamma} + Bc_1 \right). \]

2. The case \( \frac{N^2}{N + 1} \leq r \leq R_0 \) Here, from the definition (3.7), obviously we get \( \psi'_p(r) = -C\mu_p (R_0 - r)^{p - 1} \) and \( \psi''_p(r) = C\mu_p (\mu_p - 1) (R_0 - r)^{p - 2} \). Hence, it follows
\[
p'(r) \ln \left| \frac{\psi'_p(r)}{\psi''_p(r)} \right|^{p(r) - 2} \psi'(r) = -p'(r) \left| C\mu_p (R_0 - r)^{p - 1} \right|^{p(r) - 1} \ln \left| C\mu_p (R_0 - r)^{p - 1} \right|
\]
\[
\geq -p'(r) \left\{ C\mu_p R_0^{\mu_p - 1} \right\}^{p(r) - 1} \ln \left| C\mu_p R_0^{\mu_p - 1} \right|.
\]
Remember us the assumption \( 0 < C < \frac{1}{\mu_p R_0^{\mu_p - 1}} \), we conclude that
\( p'(r) \ln \left| \psi'(r) \right| \left| \psi''_p(r) \right|^{p(r) - 2} \psi'(r) > 0 \).

To end, by taking \( \mu_p = \frac{p^+}{p - 1} \), the term \( \frac{N - 1}{r} \left| \psi'(r) \right|^{p(r) - 2} \psi'(r) \) is estimated as follow:
\[
\frac{N - 1}{r} \left| \psi'(r) \right|^{p(r) - 2} \psi'(r) \geq -\frac{N - 1}{N} \left( C\mu_p (R_0 - r)^{p - 1} (\mu_p - 1) (p - 1) \right)^{-1}
\]
\[
\geq -\frac{N - 1}{N} \left( C\mu_p (R_0 - r)^{p - 1} (\mu_p - 1) (p - 1) \right)^{-1}
\]
\[
\geq -\frac{N - 1}{N} \left( C\mu_p (R_0 - r)^{p - 1} (\mu_p - 1) (p - 1) \right)^{-1}.
\]

The last term \( (p(r) - 1) \left| \psi'(r) \right|^{p(r) - 2} \psi''(r) \) is estimated as
\( (p(r) - 1) \left| \psi'(r) \right|^{p(r) - 2} \psi''(r) \geq C\mu_p (R_0 - r)^{p(r) - 1} (\mu_p - 1) (R_0 - r)^{p(r) - 1} \).

Indeed, from the definition (3.7), \( u_0 \) and \( v_0 \) are nonnegative on \( \left[ \frac{NR_0}{N + 1}, R_0 \right] \), moreover \( c(x) \) is also nonnegative, we conclude that
\[
\Delta_p(x) \psi_p(\|x\|) \geq C\mu_p (R_0 - r)^{p(r) - 1} (\mu_p - 1) (p(r) - 1) \left( \frac{N - 1}{N} \right)
\]
\[
\geq \frac{C\mu_p (R_0 - r)^{p(r) - 1} (\mu_p - 1) (p(r) - 1) \left( \frac{N - 1}{N} \right)}{N}.
\]
Finally, we have for any \( x \) satisfying \( \frac{NR_0}{N+1} \leq r = \|x\| \leq R_0 \),
\[
\Delta_p(x)\psi_p(||x||) + c(x)\psi_p(|||x|||\psi_p(||x||))^{\alpha-1}|\psi_q(||x||)|^{\beta+1} \geq 0.
\]

3. \( R_0 \leq r \) Using again (3.7), it is obvious to get
\[
\Delta_p(x)\psi_p(||x||) + c(x)\psi_p(|||x|||\psi_p(||x||))^{\alpha-1}|\psi_q(||x||)|^{\beta+1} \geq 0.
\]

\[\blacksquare\]

3.3 \( c(x) < 0 \)

Lemma 3.3 Let \( c(x) \leq 0 \). For \( p > 1 \), let \( \psi_p \) defined in \([0, R_0]\):
\[
\psi_p(r) = \begin{cases} 
-\varepsilon^{1/p^p} (R_0 - r)^{\sigma_p} & \text{if } 0 \leq r < R_0, \\
0 & \text{if } R_0 \leq r.
\end{cases}
\] (3.9)

Assume
\[
\begin{cases}
p' > 0, \text{ in } [r, R_0], \\
\exists \nu_0 > 0, \exists a > 1; \forall r > R_0 - \nu_0, \\
p(r) = 1 + o((R_0 - r)^a).
\end{cases}
\] (3.10)

\[
\sigma_p \geq \frac{p}{p' - 1},
\] (3.11)

\[
0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3),
\] (3.12)

where
\[
\varepsilon_1 = \left( \frac{1}{\sigma R_0^{\sigma_p - 1}} \right)^{p'}, \quad \varepsilon_2 = \left[ -\frac{\nu_0^{-1}(p^p - 1)(\sigma_p - 1)}{\inf_{0 \leq r \leq R_0} p'(r)} \right]^{p'}, \quad \varepsilon_3 = \left[ -\frac{\sigma_p - 1}{\inf_{0 \leq r \leq R_0} p'(r)} \right]^{p'}.
\]

then \( \psi_p \) obeys to the properties
\[
\Delta_p(x)\psi_p(||x||) + c(x)\psi_p(|||x|||\psi_p(||x||))^{\alpha-1}|\psi_q(||x||)|^{\beta+1} > 0.
\] (3.13)

\textbf{Proof.} Taking account of the definition of the function \( \psi_p \), we process similarly to the above section. That means, we claim that the properties remains valid within \( 0 \leq r \leq R_0 \) as well as to \( R_0 < r \). Indeed,
1. \( 0 \leq r < R_0 \)

After some easy calculations, we have:

\[
\psi_p^{(r)}(r) = \varepsilon^{1/p - \sigma_p} (R_0 - r)^{\sigma_p - 1},
\]

\[
\psi_p^{(r)}(r) = -\varepsilon^{1/p - \sigma_p}(\sigma_p - 1) (R_0 - r)^{\sigma_p - 2}.
\]

Using the definition of the \( p(x) \)-Laplace operator \(-\Delta_p(x)\), it follows

\[
\Delta_p(x)\psi_p(r) = \varepsilon^{\frac{p(r) - 1}{p(r)}} \left( T_{\varepsilon,1}(r) + T_2(r) + T_3(r) \right)
\]

where,

\[
T_{\varepsilon,1}(r) = -p'(r)\sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} \ln \left( \frac{1}{p(r)^{\sigma_p} (R_0 - r)^{\sigma_p - 1}} \right), \quad 0 \leq r < R_0,
\]

\[
T_{\varepsilon,1} = \begin{cases} 0 & r = R_0, \end{cases}
\]

\[
T_2(r) = \frac{N - 1}{r} \sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)},
\]

\[
T_3(r) = - (p(r) - 1)\sigma_p^{p(r) - 1}(\sigma_p - 1) (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} - 1.
\]

Let us notice from the hypotheses on \( \sigma_p \) that for any \( 0 \leq r \leq R_0 \), \( (p(r) - 1)(\sigma_p - 1) > 1 \) holds. Moreover, it is obvious that

\[
T_3(r) \geq - (p(r) - 1)\sigma_p^{p(r) - 1}(\sigma_p - 1) R_0^{(p(r) - 1)(\sigma_p - 1)}.
\]

It is also immediate to notice that

\[
T_2(r) \geq \frac{N - 1}{R_0} \sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)}.
\]

Setting \( T_\varepsilon(r) = T_{\varepsilon,1}(r) + T_2(r) + T_3(r) \), it results that

\[
T_\varepsilon(r) \geq -p'(r)\sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} \ln \left( \frac{1}{p(r)^{\sigma_p} (R_0 - r)^{\sigma_p - 1}} \right) + \frac{N - 1}{R_0} \sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} - (p(r) - 1)\sigma_p^{p(r) - 1}(\sigma_p - 1)
\]

\[
\times (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)}.
\]

Because \( R_0 - r \leq R_0 \), we also have

\[
T_\varepsilon(r) \geq -p'(r)\sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} \ln \left( \frac{1}{p(r)^{\sigma_p} R_0^{\sigma_p - 1}} \right) + \frac{N - 1}{R_0} \sigma_p^{p(r) - 1} (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)} - (p(r) - 1)\sigma_p^{p(r) - 1}(\sigma_p - 1)
\]

\[
\times (R_0 - r)^{(p(r) - 1)(\sigma_p - 1)}.
\]

14
It derives from hypotheses (3.10), for any \( r \) such that \( \nu_0 < R_0 - r \)

\[
T_\varepsilon(r) \geq \sigma_p^{p(r)-1} (R_0 - r)^{(p(r)-1)(\sigma_p-1)} \left[ - \inf_{0 \leq r \leq R_0} p'(r) \sigma_p^{p(r)-1} \ln \left( \frac{1}{\varepsilon} \sigma_p R_0^{\sigma_p-1} \right) - (p-1)\nu_0^{-1} (\sigma_p - 1) \right].
\]

Thank to the hypotheses (3.12), we have \( \ln \left( \frac{1}{\varepsilon} \sigma_p R_0^{\sigma_p-1} \right) < 0 \), consequently, the term \( T_\varepsilon \) remains positive for any \( r \) such that \( R_0 - r > \nu_0 \) and so

\[
\Delta_{p(x)} \psi_p(r) \geq 0.
\]

Now, consider the opposite case (i.e: \( R_0 - r \leq \nu_0 \)).

From the hypotheses (3.10), we have \( \frac{p(r)}{R_0 - r} < 1 \) for \( R_0 - r \leq \nu_0 \),

\[
T_\varepsilon(r) \geq -p'(r)\sigma_p^{p(r)-1} (R_0 - r)^{(p(r)-1)(\sigma_p-1)} \ln \left( \frac{1}{\varepsilon} \sigma_p R_0^{\sigma_p-1} \right) - \frac{p(r)}{R_0 - r} \sigma_p^{p(r)-1} (\sigma_p - 1) (R_0 - r)^{(p(r)-1)(\sigma_p-1)}
\]

\[
\geq \sigma_p^{p(r)-1} (R_0 - r)^{(p(r)-1)(\sigma_p-1)} \left[ -p'(r) \ln \left( \frac{1}{\varepsilon} \sigma_p R_0^{\sigma_p-1} \right) - (\sigma_p - 1) \right]
\]

\[
\geq 0.
\]

Then, as above, we retrieve \( T_\varepsilon(r) \geq 0 \) and consequently,

\[
-\Delta_{p(x)} \psi_p(r) \geq 0, \quad \forall r; \quad 0 < R_0 - r \leq \nu_0.
\]

2. \( R_0 \leq r \)

We turn to the definition of \( \psi_p \). It is obvious that

\[
\Delta_{p(x)} \psi_p(r) + c(x) \psi_p(r) \left| \psi_p(r) \right|^{\alpha-1} \left| \psi_q(r) \right|^{\beta+1} = 0.
\]

We conclude that under the assumption \( c(x) \leq 0 \), we obtain

\[
\Delta_{p(x)} \psi_p(r) + c(x) \psi_p(r) \left| \psi_p(r) \right|^{\alpha-1} \left| \psi_q(r) \right|^{\beta+1} \geq 0.
\]

This completes the proof of Lemma 3.3. ■
3.4 Construction of a supersolution

$\Omega$ is a bounded domain, so (even if it means doing a translation) there exists $B(O, R^0)$ the ball centered on $O$ and of radius $R^0$ such that $\Omega \subset B(O, R^0)$. Let $x \in \Omega$, throughout the text, we set $r = \|x\|.$

**Proposition 3.1** Assume that

\[
\frac{\alpha + 1}{p^-} + \frac{\beta + 1}{q^-} - 1 < 0.
\]

Let $(u^0, v^0)$ be defined for any $x \in \Omega$ such that $0 \leq \|x\| \leq R^0$ as follow:

\[
u^0(x) = \theta^{1/p^-} \left([R^0 - \delta r]^{\mu_0 - 1} + 1\right), \quad v^0(x) = \theta^{1/q^-} \left([R^0 - \delta r]^{\mu_0 - 1} + 1\right),
\]

where $0 < \delta < 1.$ Then, there exists $\theta^0$ depending on $\mu_0, R^0, p, q, \delta, N, c$ such that for any $\theta \geq \theta^0$,

\[
\Delta_{\mu(x)} u^0(x) + c(x)u^0(x)|u^0(x)|^{\alpha-1}|v^0(x)|^{\beta+1} \leq 0. \quad (3.14)
\]

**Proof.**

1. Some notation used in the proof of Lemma 3.1 remain the same here. Applying Lemma 3.1 with $\tilde{a} = \theta^{1/p^-}.$ We take $u^0(x) = \varphi(r),$ so we get for any $0 \leq r \leq R$

\[
\Delta_{\mu(x)} u^0(x) + c(x)u^0(x) |u^0(x)|^{\alpha-1} |v^0(x)|^{\beta+1}
\]

\[
\leq -\mathcal{K} \theta^{\frac{\mu_0 - 1}{p^-}} + \|c\|_\infty \frac{\theta^{\frac{\mu_0}{p^-}}}{r} \left(\{R^0 - \delta r\}^{\mu_0 - 1} + 1\right)^{\alpha} \frac{\beta + 1}{q^-} \left(\{R^0 - \delta r\}^{\mu_0 - 1} + 1\right)^{\beta + 1}
\]

\[
\leq -\mathcal{K} \theta^{\frac{\mu_0 - 1}{p^-}} + \|c\|_\infty \frac{\theta^{\frac{\mu_0}{p^-}} + \frac{\beta + 1}{q^-}}{r} \left(\{R^0\}^{\mu_0 - 1} + 1\right)^{\alpha} \left(\{R^0\}^{\mu_0 - 1} + 1\right)^{\beta + 1}
\]

\[
\leq -\mathcal{K} \theta^{\frac{\mu_0 - 1}{p^-}} + \mathcal{K} \theta^{\frac{\mu_0}{p^-} + \frac{\beta + 1}{q^-}},
\]

where

\[
\mathcal{K} = \|c\|_\infty \left(\{R^0\}^{\mu_0 - 1} + 1\right)^{\alpha} \left(\{R^0\}^{\mu_0 - 1} + 1\right)^{\beta + 1}.
\]

Else, it follows:

\[
\Delta_{\mu(x)} u^0(x) + c(x)u^0(x) |u^0(x)|^{\alpha-1} |v^0(x)|^{\beta+1} \leq \theta^{\frac{\mu_0}{p^-} + \frac{\beta + 1}{q^-}} \left[\mathcal{K} - \mathcal{K} \theta^{\frac{\mu_0 - 1}{p^-}} \left(\frac{\mu_0}{p^-} + \frac{\beta + 1}{q^-}\right)\right].
\]

(3.15)

Because the hypothesis $\frac{\alpha + 1}{p^-} + \frac{\beta + 1}{q^-} < 1$ holds, let us set $\theta^0 = \left(\frac{\mathcal{K}}{\mathcal{K}} \theta^{\frac{\mu_0 - 1}{p^-}} \left(\frac{\mu_0}{p^-} + \frac{\beta + 1}{q^-}\right)\right).$

For any $\theta \geq \theta^0,$ the right hand in (3.15) remains nonnegative.
4 Existence of a subsolution

Proposition 4.1 Under the hypothesis of Lemma 4.1. Let $C^-$ and $C^+$ the sets defined as follow:

\[ C^- = \{ x \in \Omega; \ c(x) < 0 \}, \ C^+ = \{ x \in \Omega; \ c(x) \geq 0 \}. \]

Let $u_0$ and $v_0$ the functions defined as follow:

\[ u_0(x) = \varphi_p(\|x\|) \mathbb{1}_{C^+}(x) + \varphi_q(\|x\|) \mathbb{1}_{C^-}(x), \ v_0(x) = \varphi_q(\|x\|) \mathbb{1}_{C^+}(x) + \varphi_q(\|x\|) \mathbb{1}_{C^-}(x). \]

the pair $(u_0, v_0)$ obeys to

\[ \Delta_p(u_0(x)) + c(x)u_0(x) |u_0(x)|^{\alpha-1} |v_0(x)|^{\beta+1} \geq 0, \ \forall x \in \Omega. \] (4.1)

Proof. This result derives to Lemma 3.2 et 3.3 by considering alternatively the cases $c(x) \geq 0$ and $c(x) < 0$. ■

4.1 A comparison result in $\Omega$

Proposition 4.2 Assume that

\[ \theta \geq \left( \frac{\theta_p^{1/p} B_p}{(R^0 - \delta d_0)^{\mu^0-1} + 1} \right)^{p^{-}}. \]

The pair $((u_0, v_0), (u^0, v^0))$ verifies the comparison result

\[ 0 < u_0(x) \leq u^0(x), \ 0 < v_0(x) \leq v^0(x), \ \forall x \in \Omega. \] (4.2)

Proof. $\Omega$ is a bounded open subset of $\mathbb{R}^N$. Thus, making the assumption the origin $O$ is in $\Omega$, there exists $x_0 \in \partial \Omega$ such that the distance $d(O, x_0) = \max_{x \in \Omega} d(O, x)$. We set $d_0 = d(O, x_0)$. We turn to Proposition 3.1 and Proposition 4.1. Particularly using the definition of $u^0$. It is clear that

\[ \inf_{x \in \Omega} u^0(x) = \varphi(d_0) = \theta^{1/p^-} \left[ (R^0 - \delta d_0)^{\mu^0-1} + 1 \right]. \]

On a other side, from Proposition 3.1, Proposition 4.1 and (3.8), we get

\[ \sup_{x \in \Omega} u_0(x) = \theta_p^{1/p^-} B_p. \]

We choose $\theta$ which as $\theta \geq \left( \frac{\theta_p^{1/p^-} B_p}{(R^0 - \delta d_0)^{\mu^0-1} + 1} \right)^{p^-}$, it is clear that

\[ \sup_{x \in \Omega} u_0(x) \leq \inf_{x \in \Omega} u^0(x). \]
Consequently, the following comparison result holds,

\[ \forall x \in \Omega, \; u_0(x) \leq u^0(x), \; v_0(x) \leq v^0(x). \]

The proof of Proposition 4.2 is complete. 

### 4.2 A sub-supersolution for (1.2)

**Proposition 4.3** Let \( \psi_p \) and \( \psi_q \) two functions given by (3.9). Assume that

1. \( (\alpha + 1)/p^- + (\beta + 1)/q^- < 1 \),
2. \( (u^0, v^0) \) and \( (u_0, v_0) \) are defined as in Proposition 3.1 and Proposition 4.1.

Then, the pair \( ((u_0, v_0), (u^0, v^0)) \) is a sub-supersolution of (1.2) in the sense of Definition 3.2.

**Proof.** We process in two step. Indeed, we distinguish the cases \( c(x) \geq 0 \) and \( c(x) < 0 \). Before starting this proof, let us note that it is obvious that \( \varphi \) is \( C^1([0, R^0]) \) and so \( u_0 \) is \( C^1(\bar{\Omega}) \). Then, \( u^0 \) and \( |\nabla u^0| \) are bounded in \( \Omega \). Consequently, \( u^0 \) belongs in \( W^{1,p(x)}(\Omega) \). We conclude that \( u^0(x) = \varphi(r) > 0 \).

Arguing similarly, we also prove that \( v^0(x) = \psi(\|x\|) \) is in \( W^{1,q(x)}(\Omega) \).

1. **Step 1:** \( c(x) \geq 0 \)

   Thanks to Proposition 4.3, we can consider \( v \) fixed as \( v_0 \leq v \leq v^0 \) for any \( x \in \Omega \).

   (a) **The case** \( 0 \leq r \leq \frac{NR_0}{N+1} \)

      i. because \( B > \frac{\|c\|_\infty}{c_1} \left( \frac{NR}{N+1} \right)^\gamma \), by considering the definition of \( u^0 \), we deduce that \( u_0(x) \geq 0 \) for any \( x \) such that \( 0 \leq \|x\| \leq \frac{NR_0}{N+1} \).

      ii. So, For any \( x \) such that \( 0 \leq \|x\| \leq \frac{NR_0}{N+1} \), one side, we have

      \[
      \Delta_{p(x)} u_0(x) + c(x) |u_0|^{\alpha-1}(x) u_0 |v|^{\beta+1} \geq \Delta_{p(x)} u_0 + c(x) (u_0)^{\alpha} (v_0)^{\beta+1} \geq 0
      \]

      and on other side

      \[
      \Delta_{p(x)} u^0 + c(x) |u^0|^{\alpha-1}(x) u^0 |v|^{\beta+1} \leq \Delta_{p(x)} u^0 + c(x) (u^0)^{\alpha} (v^0)^{\beta+1} \leq 0.
      \]
Then, assuming \( c(x) \geq 0 \), for any \( x \) such that \( 0 \leq \|x\| \leq \frac{NR_0}{N+1} \), \( u^0, u_0, v^0 \) and \( v_0 \) obey to
\[
\Delta_{p(x)} u^0 + c(x) |u^0|^{\alpha-1} u^0 |v|^{\beta+1} \leq 0 \leq \Delta_{p(x)} u_0 + c(x) |u_0|^{\alpha-1} u_0 |v|^{\beta+1}, \quad \forall v \in [v_0, v^0].
\]

(b) **The case** \( \frac{NR_0}{N+1} \leq r \leq R_0 \) From the definition (3.7), \( u_0 \) and \( v_0 \) are nonnegative on \( \left[ \frac{NR_0}{N+1}, R_0 \right] \), moreover \( c(x) \) is also nonnegative, we conclude that
\[
\Delta_{p(x)} u_0(x) + c(x) |u_0|^{\alpha-1} u_0 |v_0(x)|^{\beta+1} \geq \frac{C p(r) \mu_p^{p(r)-1} (R_0 - r) (\mu_p - 1) (p(r)-1) - N - 1}{N}
\]
Finally, we have
\[
\Delta_{p(x)} u_0(x) + c(x) |u_0|^{\alpha-1} u_0 |v_0(x)|^{\beta+1} \geq 0.
\]

Consequently, for any \( x \) satisfying \( \frac{NR_0}{N+1} \leq r = \|x\| \leq R_0 \),
\[
\Delta_{p(x)} u_0 + c(x) |u_0|^{\alpha-1} u_0 |v|^{\beta+1} \geq \Delta_{p(x)} u_0 + c(x) |u_0|^{\alpha-1} u_0 |v_0|^{\beta+1} \geq 0
\]
and then
\[
\Delta_{p(x)} u^0 + c(x) |u^0|^{\alpha-1} u^0 |v|^{\beta+1} \leq \Delta_{p(x)} u^0 + c(x) (u^0)^\alpha (v^0)^{\beta+1} \leq 0.
\]

(c) \( R_0 \leq r \) Using again (3.7), it is obvious to get
\[
\Delta_{p(x)} u_0(x) + c(x) |u_0|^{\alpha-1} u_0 |v_0(x)|^{\beta+1} \geq 0.
\]

Consequently, for any \( x \) such that \( R_0 \leq r = \|x\| \), we have
\[
\Delta_{p(x)} u_0 + c(x) |u_0|^{\alpha-1} u_0 |v|^{\beta+1} \geq 0
\]
and so, for any \( v \in [v_0, v^0] \),
\[
\Delta_{p(x)} u^0 + c(x) |u^0|^{\alpha-1} u^0 |v|^{\beta+1} \leq 0 \leq \Delta_{p(x)} u_0 + c(x) (u_0)^\alpha |v|^{\beta+1}.
\]

2. **Step 2**: \( c(x) < 0 \)

For any \( x \) such that \( 0 \leq \|x\| \leq R_0 \) and for any \( v \) fixed in \([v_0, v^0]\)
\[
T(r) \geq \frac{p'(r) \sigma_p^{p(r)-1} (R_0 - r)^{(p(r)-1)(\sigma_p - 1)} \ln \left( \frac{1}{r} \sigma_p (R_0 - r)^{\sigma_p - 1} \right)}{R_0} + \frac{N - 1}{R_0} \sigma_p^{p(r)-1} (R_0 - r)^{(p(r)-1)(\sigma_p - 1) - (p(r) - 1)\sigma_p^{p(r)-1}(\sigma_p - 1)}}
\]

\times \left( R_0 - r \right)^{(p(r)-1)(\sigma_p - 1)-1}.\]
5 A truncated system

We consider the following truncated system:

\[
\Delta_{\rho(x)} u_0 + c(x) u_0 |u_0(x)|^{\alpha-1} |v|^{\beta+1} = \varepsilon \frac{\alpha}{r} T(r) - c \varepsilon \frac{\alpha}{r} (R_0 - r)^{\sigma \alpha} |v|^{\beta+1}.
\]

Taking $\varepsilon$ as in (3.12), $T(r)$ is positive. So, assuming $c(x) \leq 0$ here, it follows that for any $v \in [v_0, v_0]$,

\[
\Delta_{\rho(x)} \psi_p (r) + c \psi_p (r) |\psi_p (r)|^{\alpha-1} |\psi_q (r)|^{\beta+1} \geq 0.
\]

By consequently, for any $x$ such that $0 \leq \|x\| \leq R_0$,

\[
\Delta_{\rho(x)} u^0 + c(x) |u^0|^{\alpha-1} u^0 |v|^{\beta+1} \leq \Delta_{\rho(x)} u_0 + c(x) (u_0)^{\alpha} |v|^{\beta+1}, \ \forall v \in [v_0, v_0].
\]

This achieves the proof of Proposition 4.3. ■

We also define the truncated energy functional:

\[
\hat{\Phi}(u, v) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\Omega} \frac{1}{q(x)} |\nabla v|^{q(x)} \, dx - \int_{\Omega} \hat{H}(x, u, v) \, dx
\]

where

\[
\hat{H}(x, u, v) = H(x, U, V) - a \int_0^u \gamma_1(x, t) \, dt - b \int_0^u \gamma_2(x, t) \, dt,
\]

\[
H(x, U, V) = c(x) |U|^{\alpha+1} |V|^{\beta+1},
\]

\[
U = u + (u_0 - u)_+ - (u - u^0)_+, \quad V = v + (v_0 - v)_+ - (v - v^0)_+,
\]

\[
\gamma_1(x, u) = -(u_0 - u)^{p-1}_+ + (u - u^0)^{p-1}_+, \quad \gamma_2(x, v) = -(v_0 - v)^{q-1}_+ + (v - v^0)^{q-1}_+.
\]

It is obvious that the associated derivative function $\hat{\Phi}'$ is defined as above:

\[
\langle \hat{\Phi}'(u, v), (w_1, w_2) \rangle_{W^{\alpha}, W} = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla w_1 \, dx - \int_{\Omega} \frac{\partial H}{\partial u}(x, u, v) w_1 \, dx
\]

\[
+ \int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \nabla w_2 \, dx - \int_{\Omega} \frac{\partial \hat{H}}{\partial v}(x, u, v) w_2 \, dx,
\]

where the notation $\langle \cdot, \cdot \rangle_{W^{\alpha}, W}$ designates the inner product between $W$ and its associated topological dual $W^*$. 

20
Lemma 5.1 Assume \( a \) and \( b \) verifying
\[
0 < a < \left( 2^{p^+-1} p^+ (K c \ diam(\Omega))^{p^-} \right)^{-1},
\]
\[
0 < b < \left( 2^{q^+-1} q^+ (K c \ diam(\Omega))^{q^-} \right)^{-1}.
\]

The truncated energy functional defined by (5.2) is bounded below on \( W^{1,p(.)}_{0}(\Omega) \times W^{1,q(.)}_{0}(\Omega) \).

Proof. Before starting, we note that
\[
\frac{\partial \hat{H}}{\partial u}(x, u, v) = \frac{\partial H}{\partial u}(x, U, V) - a \gamma_1(x, u), \quad \frac{\partial \hat{H}}{\partial v}(x, u, v) = \frac{\partial H}{\partial v}(x, U, V) - b \gamma_2(x, v).
\]
So, on
\[
\left| \frac{\partial \hat{H}}{\partial u}(x, u, v) \right| \leq |H(x, U, V)| + a \int_{0}^{\|u\|} |\gamma_1(x, u)| dt + b \int_{0}^{\|u\|} |\gamma_2(x, v)| dt
\]
\[
\leq \|c\|_{\infty} |U|^{\alpha+1} |V|^{\beta+1} + a \int_{0}^{\|u\|} 2^{p^+-1} \left( |u|^p \right) dt
\]
\[
+ b \int_{0}^{\|v\|} 2^{q^+-1} \left( |v|^q \right) dt
\]
\[
\leq \|c\|_{\infty} |U|^{\alpha+1} |V|^{\beta+1} + a 2^{p^+-1} \left( |u|^p \right) + |u|^p
\]
\[
+ b 2^{q^+-1} \left( |v|^q \right) dt.
\]
Moreover, [cf. Lemma 8.1.8 p. 250, [2]] since \( p^- \leq p(x) \), then the embedding \( L^{p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega) \) is continuous, then there exists \( K \) a positive constant such that
\[
\|u\|_{L^{p^-}(\Omega)} \leq K \|u\|_{L^{p(x)}(\Omega)}.
\]
\[
\int_{\Omega} \left| \frac{\partial \hat{H}}{\partial u}(x, u, v) \right| dx \leq K_1 + 2^{p^+-1} a C_0 \|u\|_{1,p(x)} + 2^{q^+-1} a D_0 \|v\|_{1,q(x)}
\]
\[
+ a 2^{p^+-1} \int_{\Omega} |u|^p dx + b 2^{q^+-1} \int_{\Omega} |v|^q dx
\]
\[
\leq K_1 + 2^{p^+-1} a C_0 \|u\|_{1,p(x)} + 2^{q^+-1} b D_0 \|v\|_{1,q(x)}
\]
\[
+ a 2^{p^+-1} \|u\|_{L^{p^-}(\Omega)} + b 2^{q^+-1} \|v\|_{L^{q^-}(\Omega)}.
\]
By consequently,
\[
\Phi(u) \geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{1}{q^+} \int_{\Omega} |\nabla v|^{q(x)} dx - K_1 - 2^{p^+-1} a C_0 \|u\|_{1,p(x)}
\]
\[
- 2^{q^+-1} b D_0 \|v\|_{1,q(x)} - a 2^{p^+-1} \rho_p(x) (|u|) + b 2^{q^+-1} \rho_q(x) (|v|)
\]
\[
\geq \frac{1}{p^+} \rho_p(x) (\|u\|) + \frac{1}{q^+} \rho_q(x) (\|v\|) - K_1 - 2^{p^+-1} a C_0 \|u\|_{1,p(x)}
\]
\[
- 2^{q^+-1} b D_0 \|v\|_{1,q(x)} - a 2^{p^+-1} \|u\|_{L^{p^-}(\Omega)} - b 2^{q^+-1} \|v\|_{L^{q^-}(\Omega)}.
\]
Assume $p$ and $q$ belong in $\mathcal{D}^{\log}(\Omega)$ or $p, q \in A$, we have the Poincaré inequalities (cf [2], Theo 8.2.4 p.255)

$$
\|u\|_{L^p(\Omega)} \leq c \text{diam}(\Omega) \|\nabla u\|_{L^p(\Omega)}
$$

$c = \text{cst}(n, c_{\log(p)})$ is a constant depending on $n$ and $c_{\log(p)}$. On the other part, (cf. Corolaire 8.2.5) the norms $\|\nabla u\|_{L^p(\Omega)}$ and $\|u\|_{W^{1,p}(\Omega)}$ are equivalent. (We can consider it as the same). Employing again that the injection $L^p(\Omega) \subset L^p(\Omega)$ is continuous and from above,

$$
\|u\|_{L^p(\Omega)} \leq (Kc \text{ diam}(\Omega))^{p^*} \|\nabla u\|_{L^{p^*}(\Omega)} \leq (Kc \text{ diam}(\Omega))^{p^*} \|\nabla u\|_{L^p(\Omega)}
$$

$\|\nabla u\|_{L^p(\Omega)}$ is to be small large, we have because $\|\nabla u\|_{L^p(\Omega)} \leq \rho(\|u\|)$ si $\|\nabla u\|_{L^p(\Omega)} > 1$ :

$$
\|u\|_{L^p(\Omega)} \leq (Kc \text{ diam}(\Omega))^{p^*} \|\nabla u\|_{L^{p^*}(\Omega)} \leq (Kc \text{ diam}(\Omega))^{p^*} \rho(\|u\|)
$$

Then, we obtain

$$
\hat{\Phi}(u, v) \geq \frac{1}{p^+} \rho_{p(x)}(\|u\|) + \frac{1}{q^+} \rho_{q(x)}(\|v\|) - K_1 - 2^{p^+-1}aC_0 \|u\|_{1,p(x)} - 2^{p^+-1}bD_0 \|v\|_{1,q(x)} - a2^{p^+-1}(Kc \text{ diam}(\Omega))^{p^*} \rho(\|u\|)
$$

$$
- b2^{q^+-1}(Kc \text{ diam}(\Omega))^{q^*} \rho(\|v\|).
$$

Let us fix $a$ and $b$ as in (5.4). It results that (because $\|u\|_{1,p(x)} \leq \rho(\|u\|) \leq ||u||_{1,p(x)}$ for $||u||_{1,p(x)} > 1$ for $||u||_{1,p(x)}$ and $||v||_{1,q(x)}$ large enough, we have $\hat{\Phi}(u, v)$ large enough.

We show now that $\hat{\Phi}(u, v)$ is bounded below on $W^{1,p(\cdot)}_0(\Omega) \times W^{1,q(\cdot)}_0(\Omega)$. Indeed, let us remark firstly by using the Young’ Inequalities

$$
2^{p^+-1}aC_0 \|u\|_{1,p(x)} \leq \frac{2^{p^+-1}aC_0}{\lambda} + \lambda \|u\|_{1,p(x)}
$$

and

$$
2^{q^+-1}bD_0 \|v\|_{1,q(x)} \leq \frac{2^{q^+-1}bD_0}{\delta} + \delta \|v\|_{1,q(x)}.
$$

So for $||u||_{1,p(x)} > 1$ et $||v||_{1,q(x)} > 1$, we get

$$
2^{p^+-1}aC_0 \|u\|_{1,p(x)} \leq \frac{2^{p^+-1}aC_0}{\lambda} + \lambda \rho(\|u\|)
$$
and
\[ 2^{q^+ - 1}bD_0 \|v\|_{1,q(x)} \leq \frac{2^{q^+ - 1}bD_0}{\delta} + \delta \rho(|\nabla v|). \]

It derives that
\[
\hat{\Phi}(u,v) \geq \left( \frac{1}{p^+} - 2^{p^+ - 1}a(Kc \text{diam}(\Omega))^{p^+} - \lambda \right) \rho_{p(x)}(|\nabla u|) \\
+ \left( \frac{1}{q^+} - 2^{q^+ - 1}b(Kc \text{diam}(\Omega))^{q^+} - \delta \right) \rho_{q(x)}(|\nabla v|) \\
- 2^{q^+ - 1}a C_0 \frac{\lambda}{\lambda - K_1} - 2^{q^+ - 1}b \frac{D_0}{\delta} - K_2. \tag{5.5}
\]

By consequently, chosen \( \lambda \) and \( \delta \) verifying:
\[
0 < \lambda < \frac{1}{p^+} - 2^{p^+ - 1}a(Kc \text{diam}(\Omega))^{p^+}
\]
and
\[
0 < \delta < \frac{1}{q^+} - 2^{q^+ - 1}b(Kc \text{diam}(\Omega))^{q^+},
\]
we can write
\[
\hat{\Phi}(u,v) \geq -2^{q^+ - 1}a C_0 \frac{\lambda}{\lambda - K_1} - 2^{q^+ - 1}b \frac{D_0}{\delta} - K_1 - K_2.
\]

Now, for \( \|u\|_{1,p(x)} \leq 1 \) et \( \|v\|_{1,q(x)} \leq 1 \), we have:
\[
C_0 \|u\|_{1,p(x)} \leq C_0 \text{ et } D_0 \|v\|_{1,q(x)} \leq D_0.
\]

Arguing as above, we obtain
\[
\hat{\Phi}(u,v) \geq -K_1 - 2^{q^+ - 1}a C_0 - 2^{q^+ - 1}b D_0.
\]

We can also consider the cases \( \|u\|_{1,p(x)} \geq 1 \) et \( \|v\|_{1,q(x)} \leq 1 \) or \( \|u\|_{1,p(x)} \leq 1 \) and \( \|v\|_{1,q(x)} \geq 1 \).

We conclude that \( \hat{\Phi} \) is estimated below in \( W^{1,p(\cdot),q(\cdot)} \) and so
\[
\inf_{(u,v) \in W^{1,p(\cdot),q(\cdot)}} \hat{\Phi}(u,v)
\]
is real. The proof of Lemma 5.1 is now complete. \( \blacksquare \)

Thus, we can set
\[
\theta = \inf_{(u,v) \in W^{1,p(\cdot),q(\cdot)}} \hat{\Phi}(u,v). \tag{5.6}
\]

**Lemma 5.2**

1. The minimizing problem (5.6) admits at least one solution \((u^*, v^*)\) in \( W^{1,p(\cdot),q(\cdot)} \).

2. Moreover, \((u^*, v^*)\) is at least a solution of the truncated problem (5.1).
Proof.

1. Let \((u_k, v_k)\) be a minimizing sequence for \(\Phi\), we claim that \((u_k, v_k)\) is bounded.
Assume a moment \(\|u_k\|_{1,p(x)} < 1 \) and \(\|v_k\|_{1,q(x)} < 1\), it is clear that the sequence \((u_k, v_k)\) is bounded.
Now, assume that \(\|u_k\|_{1,p(x)} > 1\) and \(\|v_k\|_{1,q(x)} > 1\). Using (5.5), we get
\[
A_{p(x)}(\|\nabla u_k\|) + B_{p(x)}(\|\nabla v_k\|) \leq \Phi(u_k, v_k) - 2^{n-1}\frac{D_0}{\delta} + K_2 + 2^{n-1}a\frac{C_0}{\lambda} + K_1.
\]
\[
A = \frac{1}{p^+} - 2^{n-1}a(Kc\text{ diam}(\Omega))^{p^-} - \lambda \text{ et } B = \frac{1}{q^+} - 2^{n-1}b(Kc\text{ diam}(\Omega))^{q^-} - \delta \text{ and from above,}
\]
\[
\|u_k\|_{1,p(x)}^p + \|v_k\|_{1,q(x)}^q \leq \min(A, B)\left[\frac{1}{\theta} + 2^{n-1}\frac{D_0}{\delta} + K_2 + 2^{n-1}a\frac{C_0}{\lambda} + K_1\right].
\]
It results that the sequences \(u_k\) and \(v_k\) are bounded in \(W_{0}^{1,p(.)}(\Omega)\) and \(W_{0}^{1,q(.)}(\Omega)\) respectively.
We can combine the cases \(\|u_k\|_{1,p(x)} > 1\) and \(\|v_k\|_{1,q(x)} < 1\) similarly \(\|u_k\|_{1,p(x)} > 1\) and \(\|v_k\|_{1,q(x)} > 1\).
We conclude that, the minimizing sequence \((u_k, v_k)\) is bounded in \(W_{0}^{1,p(.)}(\Omega) \times W_{0}^{1,q(.)}(\Omega)\). We suppose that \(1 < p^- \leq p^+ < +\infty\) and \(1 < q^- \leq q^+ < +\infty\) then in virtue of Theorem 8.1.6 [2], the spaces \(W_{0}^{1,p(.)}(\Omega)\) and \(W_{0}^{1,q(.)}(\Omega)\) are reflexive. Denote again \(u_k\) and \(v_k\) be the extracted sub-sequences which converges weakly in \(W_{0}^{1,p(.)}(\Omega)\) and \(W_{0}^{1,q(.)}(\Omega)\) respectively. Denote as \(u^*_k\) and \(v^*_k\), the weak limits of \(u_k\) and \(v_k\) in \(W_{0}^{1,p(.)}(\Omega)\) and \(W_{0}^{1,q(.)}(\Omega)\) respectively. Because \(\Phi\) is assumed w.l.s.c.i., we get \(\Phi(u^*, v^*) = \inf_{(u,v)\in W_{0}^{1,p(.)}(\Omega)} \Phi(u, v)\).

2. Since \((u^*, v^*) = \arg\min_{(u,v)\in W_{0}^{1,p(.)}(\Omega)} \Phi(u, v)\), from (5.3), the characterization holds \(\langle \Phi'(u, v), (w_1, w_2) \rangle_{W^*, W} = 0, \forall (w_1, w_2) \in W\). By taking successively \(w_1 = 0\) and \(w_2 = 0\) in (5.3), more specially, \((u^*, v^*)\) obeys to the system (5.1).

This completes the proof of the lemma 5.2. ■

In the next section, we are going to aboard the location of \((u^*, v^*)\). More precisely, we claim that \((u^*, v^*) \in [u_0, u^0] \times [v_0, v^0]\).
6 Location of \((u^*, v^*)\)

The last section is devoted by establishing the location of the solution of the truncated problem (5.1). More precisely, we have:

**Lemma 6.1** Let \([([u_0, v_0], (u^0, v^0)]\) be the pair of subsolution and supersolution given by the propositions 3.1, 4.1 and 4.3. Under the assumptions of Lemma 5.1, the pair \((u^*, v^*)\) obtained in Lemma 5.2 is as follows:

\[
u_0 \leq u^* \leq u^0, \quad v_0 \leq v^* \leq v^0.\]

**Proof.** We will only focus our proof to show that \(u_0 \leq u^*\). To start, let us associate to \((u^*, v^*)\) the pair \((U^*, V^*)\) in \(W^{1,p()}_0(\Omega) \times W^{1,q()}_0(\Omega)\) defined as follows

\[
U^* = u^* + (u_0 - u^*)_+ - (u^* - u_0)_+, \quad V^* = v^* + (v_0 - v^*)_+ - (v^* - v_0)_+. 
\]

Remember us that \((u^*, v^*)\) is a solution of the truncated system (5.1), obviously for every \(x \in \Omega:\)

\[
-\Delta p(x) u = c(x)U^* |U^*|^{\alpha - 1} |V^*|^{\beta - 1} - a\gamma_1(x, u^*)
\]

by consequently, for every \(v \in [v_0, v^0]:\)

\[
-\Delta p(x) u_0 + \Delta p(x) u^* \leq c(x) |u_0|^{\alpha - 1} v_0 |v|^{\beta + 1} - c(x) U^* |U^*|^{\alpha - 1} |V^*|^{\beta + 1} + a\gamma_1(x, u^*) .
\]

Multiply by \((u_0 - u^*)_+\) and integrate over \(\Omega\), we obtain :

\[
\int_{\Omega} \left( -\Delta p(x) u_0 + \Delta p(x) u^* \right) (u_0 - u^*) \, dx \\
\leq \int_{\Omega} c(x) \left( |u_0|^{\alpha - 1} u_0 |v|^{\beta + 1} - U^* |U^*|^{\alpha - 1} |V^*|^{\beta + 1} \right) \cdot (u_0 - u^*) \, dx \quad (6.1)
\]

+ \int_{\Omega} a\gamma_1(x, u^*) \cdot (u_0 - u^*) \, dx

where \(\Omega_+ = \{ x \in \Omega; \ u^*(x) < u_0(x) \}\). To continue, we consider the subsets of \(\Omega_+:\)

\[
\Omega_+^0 = \{ v^0 < v^* \}, \quad \Omega_+^* = \{ v_0 \leq v^* \leq v^0 \} \quad \text{and} \quad \Omega_{++} = \{ v^* < v_0 \}.
\]

It follows \(V^* = v^0 \{ \mathbb{I}_{\Omega_+^0} \} + v^* \{ \mathbb{I}_{\Omega_+^*} \} + v_0 \{ \mathbb{I}_{\Omega_{++}} \}\). By taking \(v = V^*\) in (6.1), it results because \(U^* = u_0\) in \(\Omega_+\),

\[
\int_{\Omega_+} c(x) \left( |u_0|^{\alpha - 1} u_0 |v|^{\beta + 1} - U^* |U^*|^{\alpha - 1} |V^*|^{\beta + 1} \right) \cdot (u_0 - u^*) \, dx = 0.
\]
Hence, (6.1) is reduced to

\[
\int_{\Omega_+} \left( -\Delta_{p(x)} u_0 + \Delta_{p(x)} u^* \right) (u_0 - u^*) \, dx \leq \int_{\Omega_+} a \gamma_1(x, u^*) (u_0 - u^*) \, dx. \tag{6.2}
\]

$-\Delta_{p(x)}$ is a strict monotone operator, then the left hand is positive strictly while the right hand remains negative by considering the definition of function $\gamma_1$ on $\Omega_+$. So, (6.2) becomes valid either $\Omega_+$ is empty or either $u_0 \leq u^*$ in $\Omega$. Analogously, we can also prove $u^* \leq u^0$, $v_0 \leq v^*$ and $v^* \leq v^0$.

References


